An operator $A$ is said to have a compact-normal resolvent if there exists a scalar $\lambda$ such that $(\lambda I - A)^{-1}$ is a compact and normal operator. To apply the above theorem we need to determine whether a given operator $A$ has a compact-normal resolvent.

We close this section with the following, rather interesting, remark.

Let $A$ be a closed operator in a Hilbert space $H$. We know that this does not imply boundedness of $A$. On the other hand, it is always possible to redefine the inner product on $\mathcal{D}(A)$ such that $\mathcal{D}(A)$ becomes a Hilbert space and $A$ becomes a bounded operator on $\mathcal{D}(A)$. In fact, for $x, y \in \mathcal{D}(A)$ define

$$(x, y) = (x, y) + (Ax, Ay),$$

where $(\cdot, \cdot)$ denotes the inner product in $H$. The proof of completeness of $\mathcal{D}(A)$ with respect to the norm

$$|x| = \|x\|^2 + \|Ax\|^2,$$

and the boundedness of $A$ in this new Hilbert space is left as an exercise.

4.13. Exercises

1. If $A$ is an operator on $H$ such that $Ax \perp x$ for every $x \in H$, show that $A = 0$.

2. Let $A$ be a bounded operator defined on a proper subspace of a Hilbert space $H$.
   (a) Define an operator $A_1$ on the closure $\overline{\mathcal{D}(A)}$ of the domain of $A$ by
   $$A_1x = \lim_{n \to \infty} A_nx, \quad \text{where } x_n \in \mathcal{D}(A) \text{ and } x_n \to x.$$
   Show that $A_1$ is well defined, i.e., $A_1x$ does not depend on a particular choice of the sequence $\{x_n\}$. Show that $A_1$ is a linear and bounded operator defined on $\overline{\mathcal{D}(A)}$.
   (b) Define an operator $B$ on $H$ by
   $$Bx = A_1x, \quad \text{where } x_1 \text{ is the projection of } x \text{ onto } \overline{\mathcal{D}(A)}.$$
   Show that $B$ is a bounded operator on $H$.
   (c) Show that $\|A\| = \|B\|$. Since $A = B$ on $\mathcal{D}(A)$, $B$ is an extension of $A$.

3. Let $\phi$ be a symmetric, positive, bilinear functional on a vector space $E$. Show that

$$|\phi(x, y)|^2 = \phi(x, x)\phi(y, y).$$

4. Let $\{e_n\}$ be a complete orthonormal sequence in a Hilbert space $H$ and let $\{\lambda_n\}$ be a sequence of scalars.
   (a) Show that there exists a unique operator $T$ on $E$ such that $Te_n = \lambda_ne_n$.
   (b) Show that $T$ is bounded if and only if the sequence $\{\lambda_n\}$ is bounded.
   (c) For a bounded sequence $\{\lambda_n\}$, find the norm of $T$.

5. Let $A: R^2 \to R^2$ be defined by $A[x, y] = [x + 2y, 3x + 2y]$. Find the eigenvalues and eigenvectors of $A$.

6. Let $T: C^2 \to C^2$ be defined by $T[x, y] = [x + 3y, 2x + y]$. Show that $T^* \neq T$.

7. Let $A: R^3 \to R^1$ be given by $A[x, y, z] = [3x - z, 2y - x + 3z]$. Show that $A$ is self-adjoint.

8. Compute the adjoint of each of the following operators:
   (a) $A: R^2 \to R^2$, $A[x, y, z] = [-y + z, -x + 2y, x + 2y]$. 
   (b) $B: R^2 \to R^2$, $B[x, y, z] = [x + y = z, -2x + 2z, x + y + 2y]$. 
   (c) $C: \mathcal{P}_2(R) \to \mathcal{P}_2(R)$, $C(p(x)) = x(1 + dx/dx)(p(x)) = (dx/dx)(xp(x))$, 
   where $\mathcal{P}_2(R)$ is the space of all polynomials on $R$ of degree less than or equal to 2.

9. If $A$ is a self-adjoint operator and $B$ is a bounded operator, show that $B^*AB$ is self-adjoint.

10. Prove that the representation $T = A + iB$ in Theorem 4.4.4 is unique.

11. If $A^*A + B^*B = 0$, show that $A = B = 0$.

12. Let $A$ be an operator on $H$. Show that
   (a) $A$ is anti-Hermitian if and only if $\lambda A$ is self-adjoint.
   (b) $A - A^*$ is anti-Hermitian.

13. Show that if $T$ is self-adjoint and $T \neq 0$, then $\lambda T \neq 0$ for all $\lambda \in N$.

14. Let $A$ be a self-adjoint operator. Show that
   (a) $\|A + x\|^2 = \|A\|^2 + \|x\|^2$.
   (b) The operator $U = (A - i\beta)(A + i\beta)^{-1}$ is unitary. ($U$ is called the Cayley transform of $A$).

15. The limit of a convergent sequence of self-adjoint operators is a self-adjoint operator.
(16) If $T$ is a bounded operator on $H$ with one dimensional range, show that there exist vectors $y, z \in H$ such that $Tx = (x, y)z$ for all $x \in H$. Hence show that

(a) $T^* x = (x, y)z$ for all $x \in H$,
(b) $T^* = \lambda T$, $\lambda$ is a scalar,
(c) $\|T\| = \|y\|\|z\|$,
(d) $T^* = T$ if and only if $y = az$ for some real scalar $a$.

(17) Let $A$ be a bounded self-adjoint operator on a Hilbert space $H$ such that $\|A\| < 1$. Prove that $(x, Tx) = (1 + \|A\|)\|x\|^2$ for all $x \in H$.

(18) Show that the product of isometric operators is an isometric operator.

(19) Let $\{e_n\}$ be a complete orthonormal sequence in a Hilbert space $H$. Show that an operator $A$ on $H$ is unitary if and only if $\{Ae_n\}$ is a complete orthonormal sequence in $H$.

(20) Let $\{e_n\}, n \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, be a complete orthonormal system in a Hilbert space $H$. Show that there exists a unique operator $A$ on $H$ such that $A e_n = e_{n+1}$ for all $n \in \mathbb{Z}$. Operator $A$ is called a two sided shift operator. Show that $A$ is isometric and unitary.

(21) Show that the product of two unitary operators is a unitary operator.

(22) Let $A$ be an operator on a Hilbert space. Define the exponential operator by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Show that $e^A$ is a well-defined operator. Prove the following

(a) $(e^A)^n = e^{nA}$ for any $n \in \mathbb{N}$.
(b) $e^0 = I$.
(c) $e^A$ is invertible (even if $A$ is not) and its inverse is $e^{-A}$.
(d) $e^A e^B = e^{A+B}$ for any commuting operators $A$ and $B$.
(e) If $A$ is self-adjoint, then $e^A$ is unitary.

(23) If $T$ is a normal operator on $H$ and $\lambda$ is a scalar, show that

$$\|T x - \lambda x\| = \|T x - \lambda x\|$$

for all $x \in H$.

(24) Show that if the kernel $K(x, y)$ satisfies $K(x, y) = \overline{K(y, x)}$, then for any real $\alpha$ the operator

$$T u(x) = \alpha u(x) + \int_a^b K(x, y) u(y) \, dy$$

on $L^2([a, b])$ is normal.

(25) Show that for any invertible operator $T$, the operator $T^* T$ is also invertible.

(26) If $T$ is normal, show that $T$ is invertible if and only if $T^* T$ is invertible.

(27) Prove Theorem 4.5.4.

(28) Let $T$ and $S$ be commuting operators. Show that if both $T$ and $S$ are normal, then $S + T$ and $ST$ are normal.

(29) If $T^* T = \mathcal{J}$, is it true that $TT^* = \mathcal{J}$?

(30) Let $A$, $B$, $C$, and $D$ be positive operators on a Hilbert space. Prove the following

(a) If $A \geq B$ and $C \geq D$, then $A + C \geq B + D$.
(b) If $A \geq 0$ and $a \geq 0$ ($a \in \mathbb{R}$), then $aA \geq 0$.
(c) If $A \geq B$ and $B \geq C$, then $A \geq C$.
(d) If $A \geq 0$ and $\|A\| \leq 1$, then $A \leq \mathcal{J}$.
(e) If $A > 0$, then there exists $a > 0$ ($a \in \mathbb{R}$) such that $aA < \mathcal{J}$.

(31) If $A$ is a positive operator and $B$ is a bounded operator, show that $B^* AB$ is positive.

(32) If $A$ and $B$ are positive operators and $A + B = 0$, show that $A = B = 0$.

(33) Show that for any self-adjoint operator $A$ there exist positive operators $S$ and $T$ such that $A = ST - TS$ and $ST = 0$.

(34) If $A$ is a positive definite operator, then it is invertible and its inverse is positive definite.

(35) Find operators $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T^2 = \mathcal{J}$. Which one is the positive square root of $\mathcal{J}$?

(36) Find the positive square root of the operator $T$ on $L^2([a, b])$ defined by $(Tf)(t) = g(t)\|f(t))$, where $g$ is a positive continuous function on $[a, b]$.

(37) Show that $\sqrt{A^2} = \sqrt{|A|}$.}

(38) Let $A$ and $B$ be positive operators on a Hilbert space. Show that $A^2 = B^2$ implies $A = B$.

(39) Let $A$ and $B$ be commuting positive operators. Show that $\sqrt{AB} = \sqrt{A} B$.
If $P$ is self-adjoint and $P^*$ is a projection operator, is $P$ a projection operator?

Let $T$ be a multiplication operator on $L^2([a, b])$. Find necessary and sufficient conditions for $T$ to be a projection.

Give an example of two non-commuting projection operators.

Show that $P$ is a projection if and only if $P = P^* P$.

Generalize Theorem 4.7.3 to any finite sum of projections.

If $T$ is an isometric operator, show that $T T^*$ is projection.

Show that for projections $P$ and $Q$ the operator $P + Q - PQ$ is a projection if and only if $P Q = Q P$.

Prove Theorem 4.8.2.

Show that the projection onto a closed subspace $F$ of a Hilbert space $H$ is a compact operator if and only if $F$ is finite dimensional.

Show that the operator $T: l^2 \to l^2$ defined by $T((x_n)) = (2^n x_n)$ is compact.

Show that a self-adjoint operator $T$ is compact if and only if there exists a sequence of finite dimensional operators strongly convergent to $T$.

Prove that the collection of all eigenvectors corresponding to one particular eigenvalue of an operator is a vector space.

Show that the space of all eigenvectors corresponding to one particular eigenvalue of a compact operator is finite dimensional.

Show that eigenvalues of a symmetric operator are real and eigenvectors corresponding to different eigenvalues are orthogonal.

Show that every non-zero vector is an eigenvector of the operator $A = \alpha \delta$ corresponding to the eigenvalue $\alpha$.

Show that shift operators have no eigenvalues.

Give an example of a self-adjoint operator which has no eigenvalues.

Give an example of a normal operator which has no eigenvalues.

Show that a non-zero vector $x$ is an eigenvalue of an operator $A$ if and only if $\| (Ax, x) \| \leq \| A x \| \| x \|$.

Show that if the eigenvectors of a self-adjoint operator $A$ form a complete orthogonal system and all eigenvalues are non-negative (or positive) then $A$ is positive (or strictly positive).

Prove the Spectral Theorem for the finite dimensional case: If $T: \mathbb{R}^N \to \mathbb{R}^N$ is a self-adjoint operator, then there exists an orthonormal system of vectors $\phi_1, \ldots, \phi_N \in \mathbb{R}^N$ and scalars $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ such that

$$T \phi_k = \lambda_k \phi_k, \quad k = 1, \ldots, N.$$ 

Hence the matrix corresponding to $T$ relative to the basis $\{ \phi_1, \ldots, \phi_N \}$ is

$$
\begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_N
\end{bmatrix}
$$

If $\lambda$ is an approximate eigenvalue of an operator $T$, show that $|\lambda| \leq \|T\|$.

Show that if $T$ has an approximate eigenvalue $\lambda$ such that $|\lambda| = \|T\|$, then $\sup_{\|x\|=1} |(Tx, x)| = \|T\|$.

If $\lambda$ is an approximate eigenvalue of $T$, show that $\lambda + \mu$ is an approximate eigenvalue of $T + \mu I$ and $\lambda \mu$ is an approximate eigenvalue of $\mu T$.

Show that $|\lambda| = 1$ for every approximate eigenvalue $\lambda$ of an isometric operator.

Show that every approximate eigenvalue of a self-adjoint operator is real.

Show that if $\lambda$ is an approximate eigenvalue of a normal operator $T$, then $\lambda$ is an approximate eigenvalue of $T^*$.
(68) Provide a detailed proof for Corollary 4.11.2.

(69) Prove Theorem 4.11.5.

(70) Find the Fourier transform of

\[
    f(x) = \begin{cases} 
        1 & \text{if } x \in [-a, a], \\
        0 & \text{otherwise}.
    \end{cases}
\]

(b) \( f(x) = \begin{cases} 
        1-|x|/2 & \text{if } x \in [-2, 2], \\
        0 & \text{otherwise}.
    \end{cases} \)

(71) Use Example 4.11.1(b) and Theorem 4.11.5(c) to show that

\[
    \mathcal{F}[e^{-a^2x^2}] = \frac{1}{\sqrt{2\pi a}} e^{-a^2/k^2}.
\]

(72) Show that under appropriate conditions

(a) \( \hat{f}'(k) = -i\mathcal{F}[xf(x)] \),
(b) \( \hat{f}^{(n)}(k) = (-1)^n \mathcal{F}[x^n f(x)] \).

(73) Use the Parseval relation to evaluate

(a) \( \int_{-\infty}^{\infty} \left( \frac{\sin x}{x} \right)^2 dx \),
(b) \( \int_{-\infty}^{\infty} \left( \frac{\sin x}{x} \right)^3 dx \),
(c) \( \int_{-\infty}^{\infty} \left( \frac{\sin x}{x} \right)^4 dx \).

(74) Prove that \((AB)C = A(BC)\) holds for unbounded operators.

(75) Prove that

(a) \((A + B)C = AC + BC\),
(b) \(AB + AC = A(B + C)\),
holds for unbounded operators. Give an example of operators \( A, B, C \) for which \( AB + AC \neq A(B + C) \).

(76) Show that \((A + B)^* = A^* + B^*\).

(77) Give an example of a closed operator whose domain is not a closed set.

(78) Show that \( A^{**} \) is symmetric whenever \( A \) is symmetric.

(79) If \( A \) is an operator on a Hilbert space \( H \) and there exists an operator \( B \) on \( H \) such that \( (Ax, y) = (x, By) \) for all \( x, y \in H \), show that \( A \) is bounded and \( B = A^* \).

(80) Let \( A \) be a closed operator in a Hilbert space \( H \). Prove that \( \mathcal{D}(A) \) is a Hilbert space with respect to the inner product defined by

\[
    (x, y) = (x, y) + (Ax, Ay),
\]

where \( (\cdot, \cdot) \) denotes the inner product in \( H \). Prove that \( A \) is a bounded operator on \( \mathcal{D}(A) \) with the defined inner product.