1. For the $A$ below, find $K(A)$. Also, find the solvability conditions for $Ax = b$ for a generic $b$. Find all solutions of $Ax = b$ for the $b$ below, which obeys the solvability conditions. What is the dimension of $K(A)$? What is the rank of $A$?

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

By row reduction, we get row-reduced echelon-form

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Writing the equations for $Rx = 0$ and solving for the pivots, we get

$$x_1 + x_3 = 0 \implies x_1 = -x_3$$
$$x_2 + x_3 = 0 \implies x_2 = -x_3$$
$$x_4 = 0 \implies x_4 = 0$$

so we have that the null space is composed of vectors of the form

$$\begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \\ 0 \end{pmatrix},$$

so the null space has basis $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

2. Explain what a basis is. (You may assume the reader knows what linear independence and spanning sets are.) Your explanation should include both an example and a non-example, both related to the vector space of symmetric 2x2 matrices (with real entries.) Remember in this case, the “vectors” in your vector space are actually matrices. Also, explain what the dimension of a vector space, with the given vector space as your example. (You don’t need to prove the dimension is well defined, like we did in class.)

A basis for a vector space $V$ is a set of vectors from $V$ that is linearly independent and whose span is the whole space $V$.

In the vector space of symmetric 2x2 matrices, $N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is not a basis because it doesn’t span. To prove $N$ doesn’t span, observe that any linear combination of vectors from $N$ is of the form $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which doesn’t include the vector $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

For the vector space of symmetric 2x2 matrices, this is a basis:

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
To show $B$ spans, given an arbitrary symmetric 2x2 matrix, it can be represented as a linear combination of the vectors from $B$:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To show $B$ is linearly independent, any linear combination of the vectors of $B$ is

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}$$

If this is equal to the zero vector, (which is the zero matrix) then entry (1, 1) being zero $\implies c_1 = 0$, entry (1, 2) being zero $\implies c_2 = 0$, and entry (2, 2) being zero $\implies c_3 = 0$. Since any linear combination that results in the zero vector must have all coefficients as 0, $B$ is linearly independent.

The dimension of a vector space is equal to the size of any basis. Since $B$ is a basis for the vector space of symmetric 2x2 matrices, the vector space of symmetric 2x2 matrices has dimension 3.

3. Prove that $Ax = 0$ has the same set of solutions as $Ux = 0$ and $Rx = 0$, where $U$ is the echelon form of $A$, and $R$ is the reduced row echelon form. In other words, prove that $N(A) = N(U) = N(R)$. Next, explain why $Ax = b$ does not have the same set of solutions as $Ux = b$ or $Rx = b$. (Usually, of course. It could happen.)

Since $U$ is the echelon form of $A$, it is the result of applying elementary row operations and row exchanges to $A$. Since each elementary row operation and row exchange can be represented as a matrix, $U = E_k E_{k-1} \cdots E_1 A = EA$. Since each row operation and row exchange is invertible, the product $E$ of such operations is invertible. Similarly, $R$ is a result of applying elementary row operations to $U$, so $R = FU = FE A$. For the same reasons, $F$ is invertible. Thus, $U = F^{-1} R$ and $A = E^{-1} U$.

Given any $x$ such that $Ax = 0$, $Ux = EX = E0 = 0$ and $Rx = FEAx = FE0 = 0$. Given any $x$ such that $Rx = 0$, $Ux = F^{-1} Rx = F^{-1} 0 = 0$ and $Ax = E^{-1} Ux = E^{-1} F^{-1} Rx = E^{-1} F^{-1} 0 = 0$. So to summarize, we first showed that any solution to $Ax = 0$ is a solution to $Ux = 0$, then showed that any solution to $Ux = 0$ is a solution to $Rx = 0$. Then we showed that any solution to $Rx = 0$ is a solution to $Ux = 0$, (thus showing that $Rx = 0$ and $Ux = 0$ have the same solution set) and finally that any solution to $Ux = 0$ is a solution to $Ax = 0$ (which shows that $Ux = 0$ and $Ax = 0$ have the same solution set.) By transitivity, we get that $Ax = 0$ and $Rx = 0$ have the same solution set.

Finally, the reason $Ax = b$ generally doesn’t have the same solution set as $Ux = b$ or $Rx = b$ is because generally a solution to $Ax = b$ gives a solution to $Ux = EX = Eb$. Since it is unusual for $b = Eb$, it is unusual for $Ax = b$ to have the same solution set as $Ux = b$. The same idea holds for $Rx = b$. 

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