Relaxations of Graph Partitioning and Vertex Separator Problems using Continuous Optimization

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Motivation: Graph Partitioning/Vertex Separator

- Model vertex separator problem using partition matrices
- problem is equivalent to NP-hard quadratic-quadratic program
- approximate/relax using basic eigenvalue bound and then much improved projected eigenvalue bound
- approx/relax using quadratic program and then semidefinite programming

- bounds: we follow approaches for eigenvalue and projected eigenvalue bounds in:
  Hadley, Rendl, W. 1990 [1, 5]
  Rendl, Lisser, Piacenti, (RLP) 2012 [4]
  and
  Semidefinite bounds in: W., Zhao 1996 [6].

Background/Notation

Given graph G and set sizes m

- $G = (N, E)$ edge-weighted undirected graph
  $N = \{1, 2, \ldots, n\}$ node set
  $E_{ij}, ij = 1, 2, \ldots, n$ edge weights
  $m = \begin{pmatrix} m_1 \\ \ldots \\ m_k \end{pmatrix}$ (pos. integer) set sizes, with $m^T e = n$

Set of all Partitions $P_m =$

$$\{(S_1, \ldots, S_k) : \quad S_i \subset N, \; |S_i| = m_i \; \forall i; \\ S_i \cap S_j = \emptyset \; \forall i \neq j; \; \cup_i S_i = N\}$$

Partition matrix $X \in \mathbb{R}^{n \times k}$; col. $X_{:j}$ incidence vector of S_j

$$X_{ij} = \begin{cases} 1 & \text{if } i \in S_j \\ 0 & \text{otherwise.} \end{cases}$$

Partition Matrix Constraints

linear/quadratic constraints (many are redundant)

set of: zero-one; nonnegative; linear equalities; *m*-diagonal orthogonality type; *e*-diagonal orthogonality type; and gangster constraints, respectively:

$$
\begin{array}{rcl}
\mathcal{Z} & := & \{X \in \mathbb{R}^{n\times k} : X_{ij} \in \{0,1\}, \forall ij\} \\
 & & \quad = \{X \in \mathbb{R}^{n\times k} : X_{ij}^2 = X_{ij}, \forall ij\} \\
\mathcal{N} & := & \{X \in \mathbb{R}^{n\times k} : X_{ij} \geq 0, \forall ij\} \\
\mathcal{E} & := & \{X \in \mathbb{R}^{n\times k} : Xe = e, X^T e = m\} \\
 & & \quad = \{X \in \mathbb{R}^{n\times k} : \|Xe - e\|^2 + \|X^T e - m\|^2 = 0\} \\
\mathcal{D}_O & := & \{X \in \mathbb{R}^{n\times k} : X^T X = \mathrm{Diag}\,(m)\} \\
\mathcal{D}_e & := & \{X \in \mathbb{R}^{n\times k} : \mathrm{diag}\,(XX^T) = e\} \\
\mathcal{G} & := & \{X \in \mathbb{R}^{n\times k} : X_{:i} \circ X_{:j} = 0, \forall i \neq j\}, \quad \circ \text{ Hadamard prod.}
\end{array}
$$

Equivalent Representations of Partition Matrices

The set of partition matrices in $\mathbb{R}^{nk}$, $\mathcal{M}_m =$

$$\begin{aligned}\mathcal{M}_m &= \mathcal{E} \cap \mathcal{Z} \\ &= \mathcal{E} \cap \mathcal{D}_O \cap \mathcal{N} \\ &= \mathcal{E} \cap \mathcal{D}_O \cap \mathcal{D}_e \cap \mathcal{N} \\ &= \mathcal{E} \cap \mathcal{Z} \cap \mathcal{D}_O \cap \mathcal{G} \cap \mathcal{N}\end{aligned}$$

Cuts and Separators, Objective

Cut of a partition

- $\delta(S_i, S_j)$ - set of edges between sets S_i, S_j
- $\delta(S) = \cup_{i<j<k}\delta(S_k, S_j)$ - set of edges with endpoints in distinct partition sets $S_1, \dots, S_{k-1}$
- The minimum of the cardinality $|\delta(S)|$ is denoted (objective) $\text{cut}(m) = \min\{|\delta(S)| : S \in P_m\}$

$\mathcal{G}$ has a vertex separator

graph G has a vertex separator if there exists $S \in P_m$ with $\delta(S) = \emptyset$, i.e., $\text{cut}(m) = 0$.
(see (RLP) [4], Hager, Hungerford 2013 [2] for relationship with bandwidth of graph and other applications)

Trace Representation of Cut Problem

- $B := \begin{bmatrix} ee^T - I_{k-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}^k$,
  $\mathbb{S}^k$ - $k \times k$ symm. matrices with trace inner-product.
- $A = (a_{ij})$ - adjacency matrix, $a_{ij} = \begin{cases} 1 & \text{if } E_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases}$
- $L := \mathrm{Diag}\,(Ae) - A = \sum_{ij \in E(G)} (e_i - e_j)(e_i - e_j)^T$ - Laplacian (e_i unit vectors)

Quadratic objective for cut(m)

Proposition For partition $S \in P_m$, and associated partition matrix $X \in \mathcal{M}_m$, the cardinality of the partition is

$$|\delta(S)| = \frac{1}{2} \mathrm{trace}(A - \mathrm{Diag}\,(d) XBX^T, \forall d \in \mathbb{R}^n$$

$\square$

$d = 0, Ae$ for $A, -L$, resp., were shown in RLP [4, Prop. 2].

Basic Eigenvalue Bound

Relaxed problem; (G=G(d)=A-$\mathrm{Diag}\,(d)$)

$$\begin{array}{rcll} \mathrm{cut}(m) & \geq & p^*_{eig}(m) & \\ & := & \min & \frac{1}{2}\,\mathrm{trace}\, GXBX^T \\ & & \mathrm{s.t.} & X \in \mathcal{D}_O \end{array}$$

$\mathcal{D}_O = \{X \in \mathbb{R}^{n\times k} : X^TX = M := \mathrm{Diag}\,(m)\}$
(orthogonal type cols for X)

Hoffman-Wielandt '53 [3] bound/Theorem

C, D symmetric order n, k, resp., $k \leq n$. Then
$\min \left\{\mathrm{trace}\, CXDX^T : X^TX = I_k\right\} =$
$\min \left\{\sum_{i=1}^k \lambda_i(D)\lambda_{\phi(i)}(C) : \phi : N \rightarrow \{1,\ldots,k\} \text{ is an injection}\right\}.$
minimum attained for $X = \left(p_{\phi(1)},\ldots,p_{\phi(k)}\right) Q^T$, where $p_{\phi(i)}$ normalized eigenvector to $\lambda_{\phi(k)}(C)$ and cols of $Q = \left[q_1 \quad \ldots \quad q_k\right]$ contains normalized eigenvectors q_i of $\lambda_i(D)$.

Basic Eigenvalue Bound II; with $-L$; or $G(d)$!!?

Lemma (RLP)

k-ordered eigs of $\tilde{B} := M^{1/2}BM^{1/2}$ satisfy

$$\lambda_1(\tilde{B}) \leq \lambda_2(\tilde{B}) \leq \ldots \leq \lambda_{k-2}(\tilde{B}) < \lambda_{k-1}(\tilde{B}) = 0 < \lambda_k(\tilde{B}).$$

Basic Eigenvalue Bound, apply Hoffman-Wielandt <u>Theorem</u>

Let $-\lambda_1(L) \geq -\lambda_2(L) \geq -\lambda_n(L)$ denote ordered n eigenvalues of $-L$; $-\lambda(L)$ denotes corresponding vector of eigenvalues. Pad the 0 eigenvalue of $\tilde{B}$ with further zeros to get an ordered vector of length n and denote it by $\hat{\lambda}(\tilde{B})$. Then

$$\text{cut}(m) \geq 0 > p^*_{eig} = -\lambda(L)^T\hat{\lambda}(\tilde{B})$$

Using $G(d)$

NOTE! the eigenvalue bounds depend on the choice of $\underline{d \in \mathbb{R}^n}$.

Though the function is equivalent on the feasible set of partition matrices $\mathcal{M}_m$, the values are no longer equal on the relaxed set $\mathcal{D}_O$. Unfortunately, the values are negative and not useful as a bound. We can fix $d = Ae \in \mathbb{R}^n$ and consider the bounds $\mathrm{cut}(m) \geq 0 > p^*_{eig}(A - \gamma \mathrm{Diag}\,(d)) = \frac{1}{2}\left\langle \lambda(A - \gamma \mathrm{Diag}\,(d)), \begin{pmatrix} \lambda(\tilde{B}) \\ 0 \end{pmatrix} \right\rangle_-$ for $\gamma \geq 0$.

Empirical tests on random problems show: maximum occurs for γ closer to 0 than 1; thus illustrating that the bound using $G = A$ is better than the one using $G = -L$.

Two Projected Eigenvalue Bound

Relaxed problem

$$\begin{array}{rcccl} \text{cut}(m) & \geq & p^*_{projeig}(m) & & \\ & := & \min & \frac{1}{2}\,\text{trace}\, AXBX^T & (A \text{ or } -L) \\ & & \text{s.t.} & X \in \mathcal{D}_O \cap \mathcal{E} & \end{array}$$

$\mathcal{D}_O = \{X \in \mathbb{R}^{n \times k} : X^TX = M := \mathrm{Diag}\,(m)\}$ (orthog type)

$\mathcal{E} = \{X \in \mathbb{R}^{n \times k} : Xe = e, X^Te = m\}$ (linear row/col sums)

Special Parametrization of $X \in \mathcal{E}$

$\tilde{m} = \sqrt{m}$; $n \times n$, $k \times k$ orthogonal matrices P, Q

$$P = \begin{bmatrix} \frac{1}{\sqrt{n}} e & V \end{bmatrix} \in \mathcal{O}_n, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{n}} \tilde{m} & W \end{bmatrix} \in \mathcal{O}_k. \qquad (*)$$

LEMMA: Rendl and W. 1990 [5]

Let $\tilde{M} = \mathrm{Diag}\,(\tilde{m})$. Suppose that $X \in \mathbb{R}^{n \times k}$ and $Z \in \mathbb{R}^{(n-1)\times(k-1)}$ are related by

$$X = P \begin{bmatrix} 1 & 0 \\ 0 & Z \end{bmatrix} Q^T \tilde{M}. \qquad (*)$$

Then the following holds:

1. $X \in \mathcal{E}$.
2. $X \in \mathcal{N} \Leftrightarrow VZW^T \geq -\frac{1}{n} e \tilde{m}^T$
3. $X \in \mathcal{D}_O \Leftrightarrow Z \in \mathcal{O}_{(n-1)\times(k-1)}$

Conversely, if $X \in \mathcal{E}$, then there exists Z such that the representation (*) holds. □

THEOREM

V, W as above, $\hat{X} := \frac{1}{n} e m^T \in \mathbb{R}^{n \times k}$
$\mathcal{Q} : \mathbb{R}^{(n-1)\times(k-1)} \to \mathbb{R}^{n\times k}$, $\mathcal{Q}(Z) = VZW^T\tilde{M}$
Then:
$\hat{X} \in \mathcal{E}$, and $\mathcal{Q}$ is invertible $\mathbb{R}^{(n-1)\times(k-1)} \leftrightarrow \mathcal{E} - \hat{X}$
Equivalently, $\mathcal{E}$ can be parametrized using $\hat{X} + VZW^T\tilde{M}$.

Thus, two objective functions

$\frac{1}{2}\operatorname{trace} AXBX^T =$
$\frac{1}{2}\operatorname{trace}(A\hat{X}B\hat{X}^T + (V^TAV)Z(W^T\tilde{M}B\tilde{M}W)Z^T + 2V^TA\hat{X}B\tilde{M}WZ^T)$
and
$\frac{1}{2}\operatorname{trace}((-L)XBX^T) = \frac{1}{2}\operatorname{trace}(V^T(-L)V)Z(W^T\tilde{M}B\tilde{M}W)Z^T.$

Two Projected Eigenvalue Bounds

Let
$\hat{A} = V^T A V$, $\hat{L} = V^T(-L)V$, $\hat{B} = W^T \tilde{M} B \tilde{M} W$,
$\alpha = \text{trace}\, A\hat{X}B\hat{X}^T$, $C = 2V^T A\hat{X}B\tilde{M}W$.
Then:

$$
\begin{aligned}
\text{cut}(m) \geq p^*_{projeig,A} &= \frac{1}{2}\left\{\alpha + \min_{\phi \text{ injective}} \left\{\Sigma_{i=1}^k \lambda_i(\hat{B})\lambda_{\phi(i)}(\tilde{A})\right\} + \right. \\
&\qquad \left. \min_{0 \leq \hat{X} + VZW^T\tilde{M}} \text{trace}\, CZ^T \right\} \\
&\geq p^*_{eig}
\end{aligned}
$$

$$
\begin{aligned}
\text{cut}(m) \geq p^*_{projeig,L} &= \frac{1}{2} \min_{\phi \text{ injective}} \left\{\Sigma_{i=1}^k \lambda_i(\hat{B})\lambda_{\phi(i)}(\tilde{L})\right\} \\
&\geq p^*_{eig},
\end{aligned}
$$

and note eigenvalues of $V^T L V$ are $n-1$ nonzero eigenvalues of L.

Attainment for Quadratic Terms

let $Q \in \mathbb{R}^{k-1 \times k-1}$ be orthog. with cols consisting of eigenvectors of $\hat{B}$ corresponding to eigenvalues of $\hat{B}$ in nondecreasing order;
let $P_A, P_L \in \mathbb{R}^{n-1 \times k-1}$ have orthonormal cols consisting of $k-1$ eigenvectors of $\hat{A}, \hat{L}$, respectively, corresponding to eigenvalues in nonincreasing order where the columns correspond to the largest $k-2$ followed by the smallest. Then the minimal scalar product terms in $p^*_{projeig,A}, p^*_{projeig,L}$ are attained by resp.

$$Z_A = P_A Q^T, Z_L = P_L Q^T.$$

Get two approx. solutions using $\mathcal{Q}$:

$$X_A = \hat{X} + V Z_A W^T \tilde{M}, \quad X_L = \hat{X} + V Z_L W^T \tilde{M},$$

Feasible Solutions; Upper Bounds

Using an approx. solution $\bar{X}$

Find nearest (Frobenius norm) feas. soln (use strong polytime LP)
Recall: $X \in \mathcal{E} \cap \mathcal{Z}$ implies that $Xe = e, X^T e = m$, and $X^T X = \mathrm{Diag}\,(m)$. Therefore:

$$\begin{array}{rcl} \|\bar{X} - X\|_F^2 & = & \operatorname{trace}\left(\bar{X}^T\bar{X} + X^T X - 2\bar{X}^T X\right) \\ & = & n + n + 2\operatorname{trace}\left(-\bar{X}^T X\right). \end{array}$$

Finding nearest feasible solution; a strong polytime LP

Solve the transportation problem:

$$\begin{array}{ll} \max & = \operatorname{trace} \bar{X}^T X \\ \text{s.t.} & Xe = e \\ & X^T e = m \\ & X \geq 0 \end{array}$$

Surprising: explicit solution for linear part

Lemma

Let $d \in \mathbb{R}^n$, $G = A - \mathrm{Diag}\,(d)$, $\widehat{X} = \frac{1}{n} e m^T \in \mathcal{M}_m$ *and*
$v_0 = \begin{bmatrix} (n - m_k - m_1) e_{m_1} \\ (n - m_k - m_2) e_{m_2} \\ \vdots \\ (n - m_k - m_{k-1}) e_{m_{k-1}} \\ 0 e_{m_k} \end{bmatrix}$, *where* $e_j \in \mathbb{R}^j$ *is the vector of ones of dimension* j. *Then*

$$\min_{X \in \mathcal{M}_m} \operatorname{trace} G \widehat{X} B X^T = \frac{1}{n} \langle Ge, v_0 \rangle_- .$$

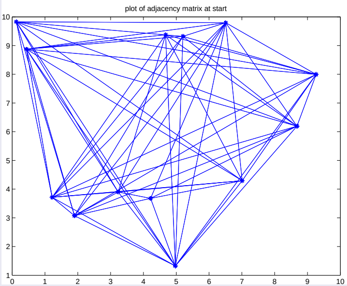
Node-Arcs for a Random Adjacency Matrix

node *i*											
1	2	3	4	5	7	8	9	10	11	12	13
2	3	4	8	9	10	11	12	13	14		
3	6	7	8	9	10	11	12	13	14		
4	7	8	9	11	13	14					
5	6	7	9	10	12	13					
6	7	9	10	12	13						
7	8	10	12	13							
8	9	10	11	12	14						
9	10	13	14								
10	11	12	14								
11	12										
12	13	14									

Table: Existing edges node *i* to node *j*

Random Ex.; Proj. Eigenvalue Lower Bound

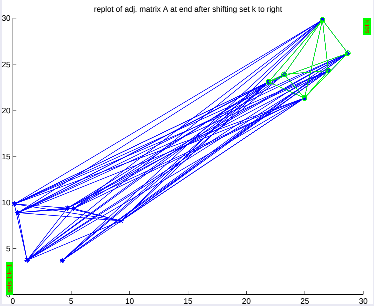
Adjacency Matrix, $m = \begin{pmatrix} 4 & 2 & 1 & 6 \end{pmatrix}, k = 4, n = 13$



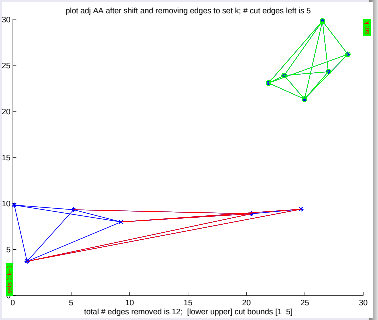
total edges: 61

Bounds, Feas. Sol., $m = \begin{pmatrix} 4 & 2 & 1 & 6 \end{pmatrix}, k = 4, n = 13$

Adjacency after shifting set k to right



Adj. after delet. edges; ([lower upper] bnds: [0.76067 5])



Quadratic-Quadratic Model

An equivalent quadratically constrained quadratic problem

$$\begin{array}{rll} \mathrm{cut}(m) \geq p^*_{SDP} = & \min & \frac{1}{2}\,\mathrm{trace}\, AXBX^T \qquad\qquad (A \text{ or } (-L)) \\ & \text{s.t.} & X \circ X = X \\ & & \|Xe - e\|^2 = 0 \\ & & \|X^t e - m\|^2 = 0 \\ & & X_{:i} \circ X_{:j} = 0 \;\forall i \neq j. \end{array}$$

where $\circ$ is the Hadamard (elementwise) product

Semidefinite Lower Bounds

Quadratic Model

We can use the various equality (quadratic) constraints in the representation and use the quadratic objective function. The Lagrangian relaxation for this quadratic-quadraitc problem is equivalent to a semidefinite program, SDP. The dual of this is the SDP relaxation. Adding redundant constraints can help.

Alternatively: directly by lifting process

linearize quadratic terms using the matrix

$$Y_X := \begin{pmatrix} 1 \\ \mathrm{vec}\,(X) \end{pmatrix} \begin{pmatrix} 1 & \mathrm{vec}\,(X)^t \end{pmatrix},$$

$\mathrm{vec}\,(X)$ is vector formed from the columns of X.
$Y_X \succeq 0$ and is rank one, the hard constraint that is relaxed.

SDP Relaxation

From direct lifting (can use A or $-L$?)

$\operatorname{trace} AXBX^T = \langle AXB, X \rangle = \mathrm{vec}\,(X)^T(\mathrm{vec}\, AXB) =$
$\mathrm{vec}\,(X)^T(B \otimes A)\mathrm{vec}\,(X) = \operatorname{trace}(B \otimes A)(\mathrm{vec}\,(X)\mathrm{vec}\,(X)^T)$

The objective function becomes $\operatorname{trace} AXBX^T = \operatorname{trace} L_A Y_X$,

$L_A := \begin{bmatrix} 0 & 0 \\ 0 & B \otimes A \end{bmatrix}$

$B \otimes A$ is the Kronecker product

Relax the rank one restriction

$$
(RGP) \qquad \begin{array}{rll} \operatorname{cut}(m) \geq p^*_{SDP} := & \min & \operatorname{trace} L_A Y \\ & \text{s.t.} & \operatorname{arrow}(Y) = e_0 \\ & & \operatorname{trace} D_1 Y = 0 \\ & & \operatorname{trace} D_2 Y = 0 \\ & & \mathcal{G}_J(Y) = 0 \\ & & Y_{00} = 1 \\ & & Y \succeq 0, \end{array}
$$

Linear Transformations

arrow operator

acting $(kn+1)\times(kn+1)$ matrix Y

$$\operatorname{arrow}(Y) := \operatorname{diag}(Y) - (0, Y_{0,1:kn})^T$$

represents the $0,1$ constraints; guarantees diagonal and 0-th row (or column) are identical;

Gangster operator $\mathcal{G}_J : \mathcal{S}_{kn+1} \to \mathcal{S}_{kn+1}$

shoots "holes" in a matrix

$$(\mathcal{G}_J(Y))_{ij} := \begin{cases} Y_{ij} & \text{if } (i,j) \text{ or } (j,i) \ \in J \\ 0 & \text{otherwise,} \end{cases}$$

$$J := \left\{(i,j) : i = (p-1)n+q,\ \ j = (r-1)n+q, \text{ for } \begin{array}{l} p < r,\ p,r \in \{1,\ldots,k\} \\ q \in \{1,\ldots,n\} \end{array} \right\}$$

represents the (Hadamard) orthogonality of the cols

Linear Transformations

The norm constraints

represented by the $(kn+1)\times(kn+1)$ matrices

$$D_1 := \begin{bmatrix} n & -e_k^t \otimes e_n^t \\ -e_k \otimes e_n & (e_k e_k^t) \otimes I_n \end{bmatrix}$$

and

$$D_2 := \begin{bmatrix} \bar{m}^t \bar{m} & -\bar{m}^t \otimes e_n^t \\ -\bar{m} \otimes e_n & I_k \otimes (e_n e_n^t) \end{bmatrix}.$$

Loss of Slater's condition

all $D_1, D_2, Y \succeq 0$,
both $\text{trace } YD_1 = 0, \text{trace } YD_2 = 0$; therefore, range of Y subset intersection of nullspaces of D_1, D_2.
feasible set of (RGP) has no strictly feasible points; implies numerical difficulties for interior-point methods.
Fix: apply facial reduction.

Facial Reduction; $Y = \hat{V} Z \hat{V}^T \in \mathbb{S}^{kn+1}$, $Z \in \mathbb{S}^{(n-1)(k-1)+1}$

$V_j \in \mathbb{R}^{j \times j-1}$

$V_j e = 0$, $V_j^T V_j = \mathrm{Diag}\,(w) \succ 0$, e.g.,

$$V_j := \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ -1 & \dots & \dots & 0 & -1 \end{bmatrix}_{j \times (j-1)} .$$

Range of $\hat{V}$ forms basis for range (any) $\hat{Y} \in \operatorname{relint} F$

$$\hat{V} := \begin{bmatrix} 1 & 0 \\ \frac{1}{n} m \otimes e_n & V_k \otimes V_n \end{bmatrix}$$

Facially Reduced SDP

Constraints for $X \in \mathcal{E}$ eliminated; $Z \in \mathbb{S}^{(n-1)(k-1)+1}$

$$
\begin{array}{rl}
\min & \operatorname{trace} \hat{V}^t L_A \hat{V} Z \\
\text{s.t.} & \operatorname{arrow}(\hat{V} Z \hat{V}^t) = 0 \\
& \mathcal{G}_J(\hat{V} Z \hat{V}^t) = 0 \\
& (\hat{V} Z \hat{V}^t)_{00} = 1 \\
& Z \succeq 0
\end{array}
$$

Slater's CQ now holds (strict feasibility).
But are we done? Are the constraints onto?

Final SDP; Slater and Onto (range of $\mathcal{G}$) Constraints

Projected onto range of gangster; $\bar{J} = J \cup (0,0)$

$$\begin{array}{ll} \min & \operatorname{trace}\left(\hat{V}^t L_A \hat{V}\right) Z \\ \text{s.t.} & \mathcal{G}_{\bar{J}}(\hat{V} Z \hat{V}^t) = \mathcal{G}_{\bar{J}}(E_{00}) \\ & Z \succeq 0 \end{array}$$

Dual program (also satisfies Slater)

$$\begin{array}{ll} \max & W_{00} \\ \text{s.t.} & \hat{V}^T \mathcal{G}_{\bar{J}}(W) \hat{V} \preceq \hat{V}^T L_A \hat{V} \end{array}$$

Numerical Tests

empirical comparisons: lower/upper bounds presented above

All the numerical tests are performed in MATLAB version R2012a on a *single* node of the *COPS* cluster at University of Waterloo. It is an SGI XE340 system, with two 2.4 GHz quad-core Intel E5620 Xeon 64-bit CPUs and 48 GB RAM, equipped with SUSE Linux Enterprise server 11 SP1.

Data				Lower bounds					Upper bounds					Rel. gap
n	k	$\|E\|$	u_0	$-L$	A	QP	SDP	DNN	$-L$	A	QP	SDP	DNN	
31	4	362	25	21	22	24	23	25	68	102	25	36	25	0.0000
18	4	86	16	13	14	15	16	16	22	35	16	19	16	0.0000
29	5	229	44	32	37	40	39	44	76	74	44	53	44	0.0000
41	5	453	91	76	84	86	86	91	159	162	101	125	102	0.0521

Table: Results for small structured graphs; comparing eigenvalue bounds using $-L, A$.

Data			Lower bounds					Upper bounds					Rel. gap
n	k	$\|E\|$	eig_{-L}	eig_A	QP	SDP	DNN	eig_{-L}	eig_A	QP	SDP	DNN	
25	4	231	53	59	64	67	71	80	79	74	75	72	0.0070
23	4	189	7	9	12	14	18	25	24	22	22	20	0.0526
32	5	379	101	112	119	123	134	152	151	141	141	137	0.0111
28	5	266	77	89	95	100	106	124	132	111	115	112	0.0230

Table: Results for small random graphs

Data				Lower bounds				Upper bounds				Rel. gap
n	k	$\|E\|$	u_0	eig_{-L}	eig_A	QP	SDP	eig_{-L}	eig_A	QP	SDP	
69	8	1077	317	249	283	290	281	516	635	328	438	0.0615
114	8	3104	834	723	785	794	758	1475	1813	834	1099	0.0246
85	8	2164	351	262	319	327	320	809	384	367	446	0.0576
116	10	3511	789	659	725	737	690	1269	2035	796	1135	0.0385
104	10	2934	605	500	546	554	529	1028	646	631	836	0.0650
78	10	1179	455	358	402	413	389	708	625	494	634	0.0893
129	12	3928	1082	879	988	1001	965	1994	1229	1233	1440	0.1022
120	12	3102	1009	833	913	926	893	1627	1278	1084	1379	0.0786
126	12	2654	1305	1049	1195	1218	1186	1767	1617	1361	1736	0.0554

Table: Results for medium-sized structured graphs

Data			Lower bounds				Upper bounds				Rel. gap
n	k	$\|E\|$	eig_{-L}	eig_A	QP	SDP	eig_{-L}	eig_A	QP	SDP	
96	8	3405	1982	2103	2126	2146	2357	2353	2354	2368	0.0460
96	8	3403	2264	2420	2439	2451	2668	2652	2658	2696	0.0394
94	8	3292	1795	1885	1910	1930	2128	2141	2092	2130	0.0403
90	10	3009	1533	1622	1649	1659	1867	1886	1850	1873	0.0544
114	10	4823	2218	2394	2443	2459	2759	2780	2725	2777	0.0513
110	10	4542	3021	3160	3185	3201	3487	3491	3484	3492	0.0423
168	12	10502	7523	7860	7894	7912	8509	8504	8494	8594	0.0355
126	12	5930	4052	4292	4318	4330	4706	4687	4672	4735	0.0380
134	12	6616	4402	4523	4557	4577	4955	5004	4963	5011	0.0397

Table: Results for medium-sized random graphs

Large instances; No SDP or DNN; A always better than $-L$

Data				Lower bounds		Upper bounds		Rel. gap
n	k	$\|E\|$	u_0	eig_{-L}	eig_A	eig_{-L}	eig_A	
2012	35	575078	361996	345251	356064	442567	377016	0.0286
1545	35	351238	210375	193295	205921	258085	219868	0.0328
1840	35	439852	313006	295171	307139	371207	375468	0.0944
1960	45	532464	346838	323526	339707	402685	355098	0.0222
2059	45	543331	393845	369313	386154	469219	483654	0.0971
2175	45	684405	419955	396363	412225	541037	581416	0.1351
2658	55	924962	651547	614044	638827	780106	665760	0.0206
2784	55	1063828	702526	664269	690186	853750	922492	0.1059
2569	55	799319	624819	586527	612605	721033	713355	0.0760

Table: Results for larger structured graphs

Data			Lower bounds		Upper bounds		Rel. gap
n	k	$\|E\|$	eig_{-L}	eig_A	eig_{-L}	eig_A	
1608	35	969450	837200	851686	875955	875521	0.0138
1827	35	1250683	1066083	1083048	1112377	1112523	0.0134
1759	35	1159454	1032413	1048350	1075600	1074945	0.0125
2250	45	1897480	1669309	1694456	1735583	1734965	0.0118
2287	45	1959760	1808192	1838114	1879230	1877722	0.0107
2594	45	2522071	2183560	2212241	2263249	2264242	0.0114
2660	55	2651856	2481928	2516160	2568521	2566434	0.0099
2715	55	2763486	2503729	2535541	2589999	2589202	0.0105
2661	55	2652743	2413321	2442960	2495530	2495115	0.0106

Table: Results for larger random graphs

n	k	$\lvert E \rvert$	density	lower	upper	Rel. gap	cpu (low)	cpu (up)
13685	68	4566914	4.88×10^{-2}	3958917	4271928	0.0380	409.4	7.1
13599	65	2282939	2.47×10^{-2}	1967979	2181778	0.0515	330.1	6.1
13795	68	1572487	1.65×10^{-2}	1314033	1495421	0.0646	316.2	7.9
13249	66	1090447	1.24×10^{-2}	832027	985375	0.0844	265.6	7.4
12425	66	767961	9.95×10^{-3}	589226	710093	0.0930	253.2	6.0
13913	66	803074	8.30×10^{-3}	591486	726783	0.1026	304.9	7.1
14144	65	711936	7.12×10^{-3}	543017	666721	0.1023	274.4	7.1
13667	67	581930	6.23×10^{-3}	427464	538291	0.1148	254.9	6.5
12821	68	455329	5.54×10^{-3}	329902	422417	0.1230	244.5	7.4
12191	69	370595	4.99×10^{-3}	262521	343426	0.1335	211.1	6.3

Table: Large scale random graphs; imax 400; $k \in [65, 70]$, using V_0

n	k	$\|E\|$	density	lower	upper	Rel. gap	cpu (low)	cpu (up)
14680	69	5254939	4.88×10^{-2}	4586083	4955524	0.0387	262.9	6.4
14464	65	2583109	2.47×10^{-2}	2133187	2397098	0.0583	135.5	6.0
14974	69	1852955	1.65×10^{-2}	1555718	1776249	0.0662	98.2	6.9
13769	65	1177579	1.24×10^{-2}	956260	1124729	0.0810	44.4	5.9
13852	69	954632	9.95×10^{-3}	775437	924265	0.0876	51.3	6.0
12516	65	650028	8.30×10^{-3}	475477	598372	0.1144	34.0	4.3
13525	66	651025	7.12×10^{-3}	508512	630663	0.1072	33.3	5.8
13622	66	578111	6.23×10^{-3}	414786	535755	0.1273	34.6	6.0
13004	65	468437	5.54×10^{-3}	328925	434795	0.1386	29.1	5.2
14659	69	535899	4.99×10^{-3}	380571	501082	0.1367	27.2	5.9

Table: Large scale random graphs; imax 400; $k \in [65, 70]$, using V_1

n	k	$\lvert E\rvert$	density	lower	upper	Rel. gap	cpu (low)	cpu (up)
22840	80	12721604	4.88×10^{-2}	11548587	12262688	0.0300	782.4	12.5
16076	77	3190788	2.47×10^{-2}	2754650	3053622	0.0515	199.1	8.9
20635	77	3519170	1.65×10^{-2}	2916188	3287657	0.0599	228.5	10.1
19408	79	2339682	1.24×10^{-2}	1989278	2272340	0.0664	147.3	10.6
17572	76	1536161	9.95×10^{-3}	1188933	1417085	0.0875	83.6	9.0
18211	80	1376087	8.30×10^{-3}	1127696	1336407	0.0847	90.7	11.2
21041	80	1575333	7.12×10^{-3}	1232501	1482463	0.0921	93.6	10.5
20661	77	1329856	6.23×10^{-3}	1023056	1251437	0.1004	74.5	11.8
19967	77	1104350	5.54×10^{-3}	831335	1035126	0.1092	74.0	9.6
20839	78	1082982	4.99×10^{-3}	831672	1034104	0.1085	73.9	11.0

Table: Large scale random graphs; imax 500; $k \in [75, 80]$, using V_1

Summary

- Model NP hard problems using quadratic-quadratic models
- First Relaxations lead to eigenvalue problems and efficient lower and upper bounds
- Lagrangian Relaxation leads to a quadratic bound and an SDP bound
- The Slater condition typically fails for SDP relaxations (facial reduction is needed for stability)
- empirical results show strength of bounds
- (strong) projected eigenvalue upper/lower bounds can be found for huge problems

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Thanks for your attention!

Relaxations of Graph Partitioning and Vertex Separator Problems using Continuous Optimization

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