Numerical Optimization of Eigenvalues of Hermitian Matrix-Valued Functions

Emre Mengi

Department of Mathematics
Koç University
İstanbul, Turkey

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Emre Yıldırım, Mustafa Kılıç, Karl Meerbergen, Wim Michiels, Raul Van Beeumen

Outline

- Unconstrained Eigenvalue Optimization
  - Use of Global Under-estimator Functions
  - Global Under-estimator Functions for Eigenvalue Functions
  - Optimization of Global Under-estimator Functions
  - Numerical Examples
- Optimization with an Eigenvalue Constraint

Unconstrained Eigenvalue Optimization

Problem

Given a matrix-valued function $\mathcal{A}(\omega) : \mathbb{R}^d \to \mathbb{C}^{n \times n}$, that is

- analytic on $\mathbb{R}^d$, and
- such that $\mathcal{A}(\omega)^* = \mathcal{A}(\omega) \;\; \forall \omega \in \mathbb{R}^d$.

Unconstrained Global Eigenvalue Optimization

$$\min_{\omega \in \mathcal{B}_d} \lambda_j(\mathcal{A}(\omega)) \quad \text{or} \quad \max_{\omega \in \mathcal{B}_d} \lambda_j(\mathcal{A}(\omega))$$

$$\mathcal{B}_d := \left\{ \omega \in \mathbb{R}^d \mid \omega_j \in \left[\omega_j^{(\ell)}, \omega_j^{(u)} \right] \quad j = 1, \ldots, d \right\}$$

$\lambda_j(\cdot)$ - jth largest eigenvalue

Applications

H_∞-norm of a Linear System

The transfer function of the system

$$x'(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t).$$

given by $\mathcal{H}(s) = C(sI - A)^{-1}B + D$ satisfies $Y(s) = \mathcal{H}(s)U(s)$.

Eigenvalue Optimization Characterization

$$\sup_{\omega \in \mathbb{R}} \sigma_1\left(\mathcal{H}(\omega)\right) \quad \text{where} \quad \mathcal{H}(\omega) := C(\omega i I - A)^{-1}B + D$$

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Eigenvalue Optimization Characterization

$$\sup_{\omega \in \mathbb{R}} \lambda_1 \left(\mathcal{H}(\omega)^* \mathcal{H}(\omega)\right) \quad \text{where} \quad \mathcal{H}(\omega) := C(\omega i I - A)^{-1}B + D$$

Applications

Minimization of the Maximum Eigenvalue

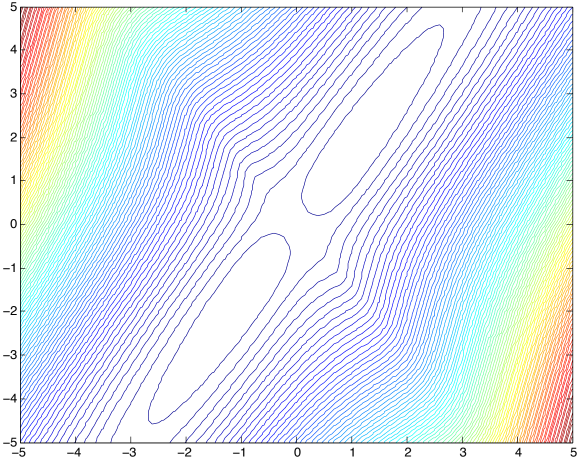
$$\min_{\omega \in \mathbb{R}^d} \lambda_1 \left(\mathcal{A}(\omega) \right)$$

where $\mathcal{A}(\omega)$ is Hermitian and depends on ω analytically.

Structural design, e.g., designing a strongest column subject to volume consraints

Robust control theory, e.g., optimizing stability

Applications



$$\lambda_1\left(\begin{bmatrix} -2+0.3\omega_1^2-1.3\omega_2^2-4\omega_1\omega_2 & 1+0.5\omega_1^2 & -3+2\omega_1\omega_2 \\ 1+0.5\omega_1^2 & -2+0.3\omega_1^2-0.3\omega_2^2-5\omega_1\omega_2 & 1.2+\omega_2^2 \\ -3+2\omega_1\omega_2 & 1.2+\omega_2^2 & -2-0.7\omega_1^2-0.3\omega_2^2-4\omega_1\omega_2 \end{bmatrix}\right)$$

Applications

Distance Problems in Numerical Linear Algebra

Given a matrix polynomial

$$P(\omega) := \sum_{j=1}^{k} \omega^j A_j$$

for fixed $A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$, consider

$$\min \left\{ \|\Delta\|_2 \mid P(\omega) + \Delta \ \text{ has a multiple eigenvalue} \right\}.$$

Eigenvalue Optimization Characterization, M.-Karow

$$\min_{\omega \in \mathbb{C}} \max_{\gamma \in \mathbb{R}} \sigma_{2n-1} \left(\begin{bmatrix} P(\omega) & \gamma P'(\omega) \\ 0 & P(\omega) \end{bmatrix} \right)$$

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Unconstrained Eigenvalue Optimization, Use of Support Functions

Support Functions

Definition (Support Functions)

A function $s(\cdot\,;\tilde{\omega}) : \mathbb{R}^d \to \mathbb{R}$ is said to be a support function for $f : \mathbb{R}^d \to \mathbb{R}$ about $\tilde{\omega} \in \mathbb{R}^d$ if

- $s(\omega;\tilde{\omega}) \leq f(\omega) \quad \forall \omega \in \mathbb{R}^d$, and
- $s(\tilde{\omega};\tilde{\omega}) = f(\tilde{\omega})$

Support function ideas have been utilized widely for global optimization. [Piyavskii, 1972] [Shubert, 1972] [Breiman&Cutler, 1993] [Jones&Perttunen&Stuckman, 1993] [Gergel] [Kvasov&Sergeyev]

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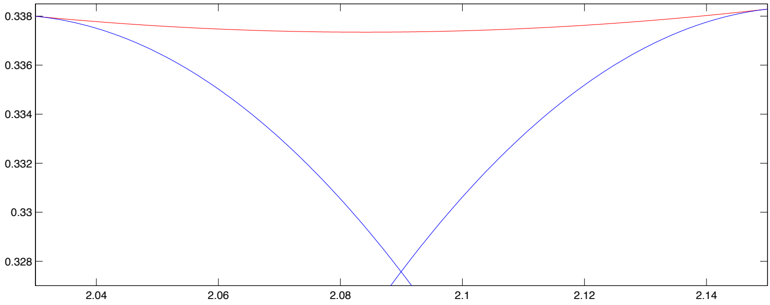
A Generic Support Function Based Algorithm

$\lambda(\omega) := \sigma_n(\omega i I - A)$ over $\mathcal{B} := [2, 2.15]$.



$$\omega_2 = \arg\min_{\omega \in \mathcal{B}} s(\omega)$$
$$\text{where } s(\omega) := \max(s(\omega; \omega_0), s(\omega; \omega_1))$$

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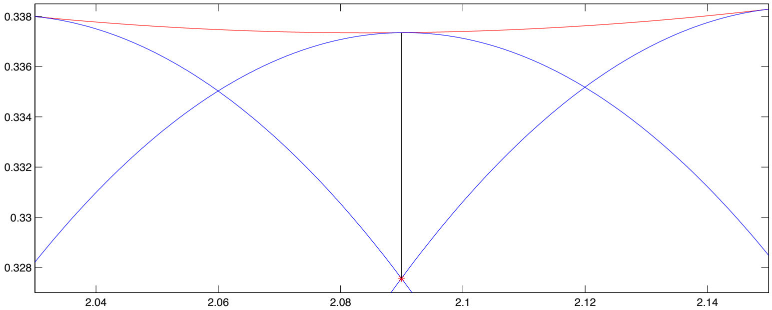

$$\omega_3 = \arg\min_{\omega \in \mathcal{B}} s(\omega)$$
where $s(\omega) := \max(s(\omega; \omega_0), s(\omega; \omega_1), s(\omega; \omega_2))$

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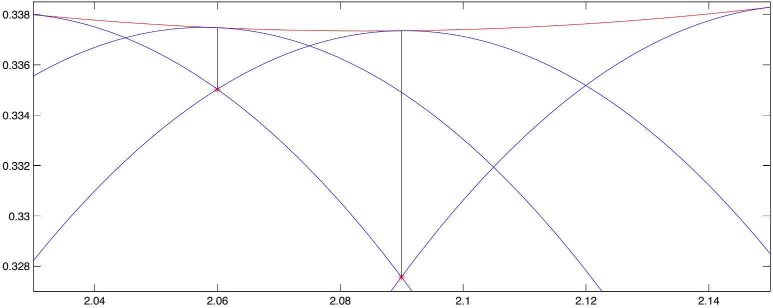

$$\omega_4 = \arg\min_{\omega \in \mathcal{B}} s(\omega)$$
where $s(\omega) := \max_{k=0,\dots,3} s(\omega; \omega_k)$

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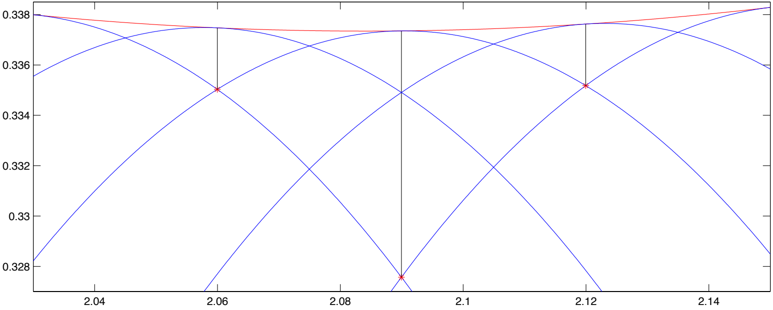

$$\omega_5 = \arg\min_{x \in \mathcal{B}} s(\omega)$$
$$\text{where } s(\omega) := \max_{k=0,\ldots,4} s(\omega; \omega_k)$$

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Letting $\lambda(\omega) := \lambda_j(\mathcal{A}(\omega))$

```
construct s(ω; ω_0) for any ω_0 ∈ B_d.
ω_1 ← arg min_{ω∈B_d} (s(ω) := s(ω; ω_0)).
l_1 ← s(ω_1; ω_0), u_1 ← min(λ(ω_0), λ(ω_1)), p ← 1.
While u_p − l_p > ε do
loop
    construct s(ω; ω_p).
    ω_{p+1} ← arg min_{ω∈B_d} (s(ω) := max_{k=0,...,p} s(ω; ω_k)).
    l_{p+1} ← s(ω_{p+1}), u_{p+1} ← min(u_p, λ(ω_{p+1})), p ← p + 1.
end loop
Output: l_p, u_p.
```

Note

$$\ell = \min_{\omega \in \mathcal{B}_d} s(\omega) \;\leq\; \min_{\omega \in \mathcal{B}_d} \lambda(\omega) \;\leq\; \min\left(\lambda(\omega_0), \ldots, \lambda(\omega_p)\right) = u$$

A Generic Support Function Based Algorithm

Letting $\lambda(\omega) := \lambda_j(\mathcal{A}(\omega))$

1: construct $s(\omega; \omega_0)$ for any $\omega_0 \in \mathcal{B}_d$.
2: $\omega_1 \leftarrow \arg\min_{\omega \in \mathcal{B}_d} \; (s(\omega) := s(\omega; \omega_0))$.
3: $l_1 \leftarrow s(\omega_1; \omega_0)$, $u_1 \leftarrow \min(\lambda(\omega_0), \lambda(\omega_1))$, $p \leftarrow 1$.
4: While $u_p - l_p > \epsilon$ do
5: **loop**
6: construct $s(\omega; \omega_p)$.
7: $\omega_{p+1} \leftarrow \arg\min_{\omega \in \mathcal{B}_d} \; (s(\omega) := \max_{k=0,\dots,p} s(\omega; \omega_k))$.
8: $l_{p+1} \leftarrow s(\omega_{p+1})$, $u_{p+1} \leftarrow \min(u_p, \lambda(\omega_{p+1}))$, $p \leftarrow p + 1$.
9: **end loop**
10: **Output:** l_p, u_p.

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Unconstrained Eigenvalue Optimization, Support Functions for Eigenvalue Functions

Analyticity Result (Univariate Case)

Theorem (Rellich, 1937)

Let $\mathcal{A}(\omega) : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ be a Hermitian matrix-valued function that depends on ω analytically.

(i) *The roots of the characteristic polynomial of $\mathcal{A}(\omega)$ can be arranged so that each root $\tilde{\lambda}_j(\mathcal{A}(\omega))$ is analytic w.r.t. ω.*

(ii) *There exists an analytic eigenvector $v_j(\mathcal{A}(\omega))$ associated with $\tilde{\lambda}_j(\mathcal{A}(\omega))$ for $j = 1, \dots, n$ such that $\{v_1(\mathcal{A}(\omega)), \dots, v_n(\mathcal{A}(\omega))\}$ is orthonormal.*

Short-hands

$\tilde{\lambda}_j(\omega) := \tilde{\lambda}_j(\mathcal{A}(\omega))$ *analytic eigenvalues*

$\lambda_j(\omega) := \lambda_j(\mathcal{A}(\omega))$ *sorted eigenvalues*

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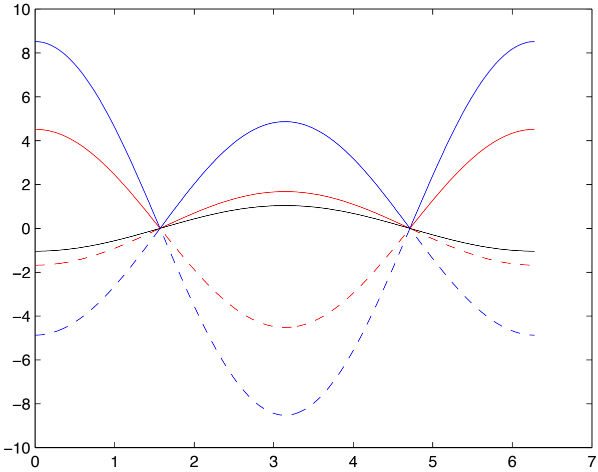
Anayticity Result (Univariate Case)

$\lambda_j(\omega)$ is analytic everywhere except those ω where $\lambda_j(\omega)$ is not simple.

If $\lambda_j(\omega)$ is not simple, it is piece-wise analytic and continuous.



Analyticity (Multivariate Case)

For a multivariate Hermitian function $\mathcal{A}(\omega) : \mathbb{R}^d \rightarrow \mathbb{C}^{n \times n}$ the eigenvalues are not analytic no matter how they are ordered.

But there is an ordering such that each eigenvalue is analytic over any line in $\mathbb{R}^d$ (Rellich's result).

The analyticity of $\tilde{\lambda}_j(\omega)$ over lines in $\mathbb{R}^d$ implies its twice differentiability, thus the existence of a γ such that

$$\lambda_{\min}\left[\nabla^2 \tilde{\lambda}_j(\omega)\right] \geq \gamma$$

for all $\omega \in \mathcal{B}_d$ for $j = 1, \ldots, n$.

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Support Functions for Extreme Eigenvalues

$$\min_{\omega \in \mathcal{B}_d} \lambda_1(\omega)$$

Theorem (Quadratic Support Functions, MYK)

Suppose $\tilde{\omega}$ *is such that* $\lambda_1(\tilde{\omega})$ *is simple, and* γ *satisfies*

$$\lambda_{\min}\left[\nabla^2 \lambda_1(\omega)\right] \geq \gamma$$

for all $\omega \in \mathcal{B}_d$ *such that* $\lambda_1(\omega)$ *is simple. Then*

$$s(\omega; \tilde{\omega}) := \lambda_1 + \nabla \lambda_1^T (\omega - \tilde{\omega}) + \frac{\gamma}{2} \|\omega - \tilde{\omega}\|^2$$

is a support function for $\lambda_1(\omega)$*, where* $\lambda_1 := \lambda_1(\tilde{\omega})$ *and* $\nabla \lambda_1 := \nabla \lambda_1(\tilde{\omega})$*.*

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is a support function for $\lambda_1(\omega)$, where $\lambda_1 := \lambda_1(\tilde{\omega})$ and $\nabla \lambda_1 := \nabla \lambda_1(\tilde{\omega})$.

Support Functions for Extreme Eigenvalues

Theorem (Deducing γ analytically, MYK)

The eigenvalue function $\lambda_1(\omega) := \lambda_1(\mathcal{A}(\omega))$ satisfies

$$\lambda_{\min}\left[\nabla^2 \lambda_1(\omega)\right] \geq \lambda_{\min}\left[\nabla^2 \mathcal{A}(\omega)\right]$$

for each ω such that $\lambda_1(\omega)$ is simple, where

$$\nabla^2 \mathcal{A}(\omega) := \begin{bmatrix} \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_1^2} & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_1 \partial \omega_2} & \cdots & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_1 \partial \omega_d} \\ \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_2 \partial \omega_1} & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_2^2} & \cdots & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_2 \partial \omega_d} \\ & & \ddots & \\ \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_d \partial \omega_1} & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_d \omega_2} & \cdots & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_d^2} \end{bmatrix}.$$

Outline of a proof

There are analytic formulas for the second derivatives of the form

$$\frac{\partial^2 \lambda_1(\mathcal{A}(\omega))}{\partial \omega_k \, \partial \omega_\ell} = v_1^*(\omega) \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_k \, \partial \omega_\ell} v_1(\omega) +$$

$$2 \cdot \Re \left[\sum_{m=2}^{n} \frac{1}{\lambda_1(\mathcal{A}(\omega)) - \lambda_m(\mathcal{A}(\omega))} \left(v_1(\omega)^* \frac{\partial \mathcal{A}(\omega)}{\partial \omega_k} v_m(\omega) \right) \left(v_m(\omega)^* \frac{\partial \mathcal{A}(\omega)}{\partial \omega_\ell} v_1(\omega) \right) \right].$$

For the Hessian, this yields

$$\nabla^2 \lambda_1(\mathcal{A}(\omega)) = \mathcal{H}(\omega) + 2 \cdot \sum_{m=2}^{n} \frac{1}{\lambda_1(\mathcal{A}(\omega)) - \lambda_m(\mathcal{A}(\omega))} \Re(\mathcal{H}^{(m)}(\omega))$$

where $\mathcal{H}(\omega)$ and $\mathcal{H}^{(m)}(\omega)$ are such that their (k, ℓ) entries are given by

$$v_1^*(\omega) \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_k \, \partial \omega_\ell} v_1(\omega) \quad \text{and} \quad \left(v_1(\omega)^* \frac{\partial \mathcal{A}(\omega)}{\partial \omega_k} v_m(\omega) \right) \left(v_m(\omega)^* \frac{\partial \mathcal{A}(\omega)}{\partial \omega_\ell} v_1(\omega) \right),$$

respectively. It can be shown that $\mathcal{H}^{(m)}(\omega)$ and $\Re(\mathcal{H}^{(m)}(\omega))$ are positive definite. Thus $\lambda_{\min}\left[\nabla^2 \lambda_1(\mathcal{A}(\omega))\right] \geq \lambda_{\min}\left[\mathcal{H}(\omega)\right] \geq \lambda_{\min}\left[\nabla^2 \mathcal{A}(\omega)\right]$.

Outline of a proof

There are analytic formulas for the second derivatives of the form

$$\frac{\partial^2 \lambda_1(\mathcal{A}(\omega))}{\partial \omega_k \, \partial \omega_\ell} = v_1^*(\omega) \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_k \, \partial \omega_\ell} v_1(\omega) +$$

$$2 \cdot \Re \left[\sum_{m=2}^{n} \frac{1}{\lambda_1(\mathcal{A}(\omega)) - \lambda_m(\mathcal{A}(\omega))} \left(v_1(\omega)^* \frac{\partial \mathcal{A}(\omega)}{\partial \omega_k} v_m(\omega) \right) \left(v_m(\omega)^* \frac{\partial \mathcal{A}(\omega)}{\partial \omega_\ell} v_1(\omega) \right) \right].$$

For the Hessian, this yields

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Support Functions for Extreme Eigenvalues

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Example:

Consider $\mathcal{A} : \mathbb{R} \to \mathbb{R}^{n \times n}$, $\mathcal{A}(\omega) := A_0 + \omega A_1 + \omega^2 A_2$ *and* $\lambda_1(\omega) := \lambda_1(\mathcal{A}(\omega))$ *for given symmetric* $A_0, A_1, A_2 \in \mathbb{R}^{n \times n}$*. Then*

$$s(\omega; \tilde{\omega}) := \lambda_1 + \lambda_1'(\omega - \tilde{\omega}) + \frac{\gamma}{2}(\omega - \tilde{\omega})^2$$

is a support function with $\lambda_1 := \lambda_1(\mathcal{A}(\tilde{\omega}))$, $\lambda_1' := \lambda_1'(\mathcal{A}(\tilde{\omega})) = v_1(\tilde{\omega})^T (A_1 + 2\omega A_2) v_1(\tilde{\omega})$ *and* $\gamma = 2\lambda_{\min}(A_2)$.

All of this (i.e., setting-up a support function, deducing γ analytically) generalize for $\sum_{j=1}^{k} c_j \lambda_j(\omega)$ with $c_1 \geq \cdots \geq c_k \geq 0$.

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Unconstrained Eigenvalue Optimization, Optimization of Support Functions

Minimization of Support Functions

construct $s(\omega; \omega_0)$ for any $\omega_0 \in \mathcal{B}_d$.
$\omega_1 \leftarrow \arg\min_{\omega \in \mathcal{B}_d} \ (s(\omega) := s(\omega; \omega_0))$.
$l_1 \leftarrow s(\omega_1; \omega_0)$, $u_1 \leftarrow \min(\lambda(\omega_0), \lambda(\omega_1))$, $p \leftarrow 1$.
While $u_p - l_p > \epsilon$ do
loop
    construct $s(\omega; \omega_p)$.
    $\omega_{p+1} \leftarrow \arg\min_{\omega \in \mathcal{B}_d} \ (s(\omega) := \max_{k=0,\dots,p} s(\omega; \omega_k))$.
    $l_{p+1} \leftarrow s(\omega_{p+1})$, $u_{p+1} \leftarrow \min(u_p, \lambda(\omega_{p+1}))$, $p \leftarrow p + 1$.
end loop
Output: l_p, u_p.

Minimizing Maximal Quadratic Support Function

$$\arg\min_{\omega \in \mathcal{B}_d} \left(s(\omega) := \max_{k=0,\ldots,p} s(\omega; \omega_k) \right)$$

where $$s(\omega; \omega_k) := \lambda_1(\mathcal{A}(\omega_k)) + \nabla\lambda_1(\mathcal{A}(\omega_k))^T(\omega - \omega_k) + \frac{\gamma}{2}\|\omega - \omega_k\|^2$$

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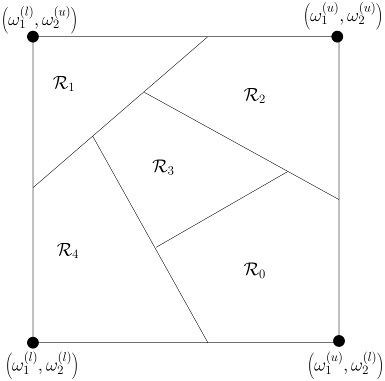

Split $\mathcal{B}_d$ into subregions. In subregion $\mathcal{R}_k$, $s(\omega;\omega_k) \geq s(\omega;\omega_j)\ \forall j \neq k$.

Minimizing Maximal Quadratic Support Functions

Solve the quadratic program (QP) for $k = 0, \dots, p$.

$$\begin{array}{ll} \min_{\omega \in \mathbb{R}^d} & s(\omega; \omega_k) \\ \text{subject to} & s(\omega; \omega_k) \geq s(\omega; \omega_j), \quad j \neq k \\ & \omega \in \mathcal{B}_d \end{array}$$

The constraints $s(\omega; \omega_k) \geq s(\omega; \omega_j)$ are linear. Thus the subproblems are quadratic programs.

Solution for each subproblem is attained at one of the vertices of its feasible region.

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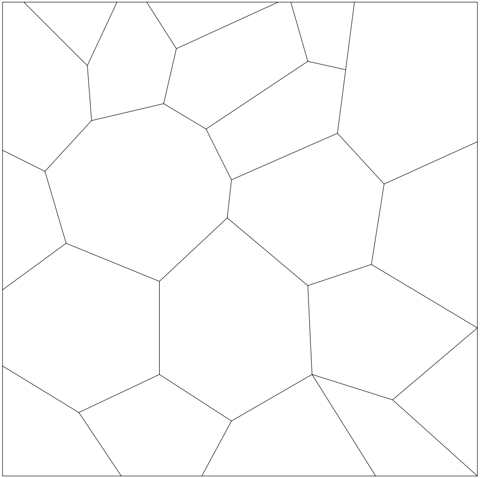
When $s(\omega; \omega_{p+1})$ is introduced, a new polytope $\mathcal{R}_{p+1}$ appears.



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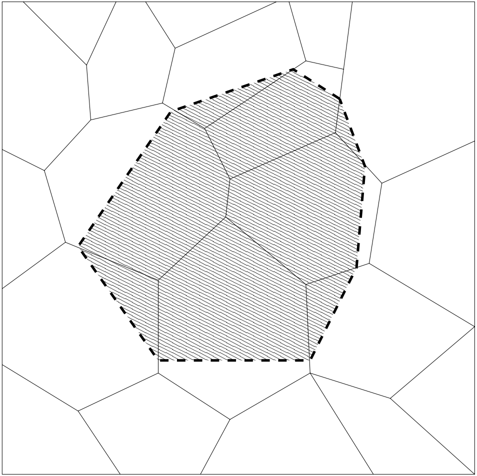

$\mathcal{D}_{p+1}$ (Dead Vertices)

set of points that were vertices before, no longer vertices.

Minimizing Maximal Quadratic Support Function

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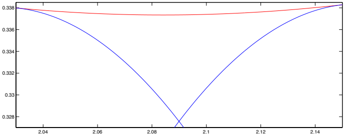

For a vertex v at step p

$$v \in \mathcal{D}_{p+1} \iff s(v; \omega_{p+1}) > \max_{k=0,\ldots,p} s(v; \omega_k).$$

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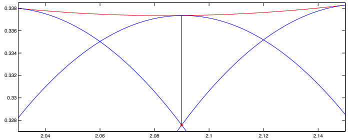

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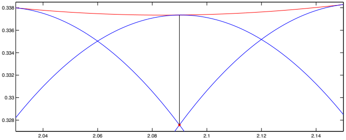

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Theorem (Dead Vertices, Breiman-Cutler)

The set $\mathcal{D}_{p+1}$ is a connected graph.

New vertices on $\mathcal{R}_{p+1}$

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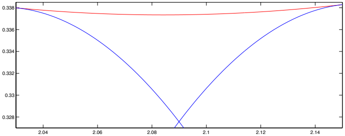
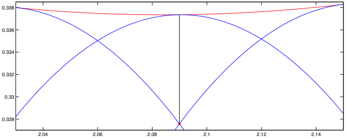


A new vertex forms between each dead vertex, and each vertex that is not dead and adjacent to the dead vertex.

New vertices on $\mathcal{R}_{p+1}$



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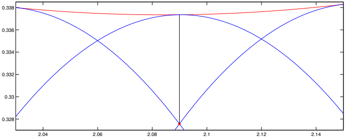
Minimizing Maximal Quadratic Support Function

New vertices on $\mathcal{R}_{p+1}$



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Theorem (Vertices of the New Polytope, Breiman-Cutler)

A vertex v on $\mathcal{C}_{p+1}$ is either on the corner of the box (only box constraints are active), or

$v = \alpha v_j + (1-\alpha) v_k$ *where* $v_j \in \mathcal{D}_{p+1}$, v_k *is an alive vertex, and*

$$\alpha = \frac{s(v_k;\omega_{p+1}) - s(v_k)}{\left[s(v_k;\omega_{p+1}) - s(v_k)\right] - \left[s(v_j;\omega_{p+1}) - s(v_j)\right]}$$

with

$s(v_k) = \max_{\ell=0,\dots,p} s(v_k;\omega_\ell)$, $s(v_j) = \max_{\ell=0,\dots,p} s(v_j;\omega_\ell)$

Data Structures

A heap to keep vertices $\{v_k\}$ sorted based on

$$\max_{\ell=0,\ldots,p} s(v_k; \omega_\ell)$$

Adjacency lists for edges

Stack to determine dead vertices

Convergence

Theorem (Convergence, MYK)

Let $\lambda_j : \mathbb{R}^d \to \mathbb{R}$ be a function analytic along every line in $\mathbb{R}^d$. Then every limit point of the sequence of iterates generated by the support-based algorithm is a global minimizer of λ_j over the box $\mathcal{B}_d$.

The rate of convergence appears *linear*.

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Unconstrained Eigenvalue Optimization, Numerical Examples

Minimizing Largest Eigenvalue

$$\min_{\omega_1,\omega_2} \lambda_1\left(\mathcal{A}(\omega_1,\omega_2)\right)$$

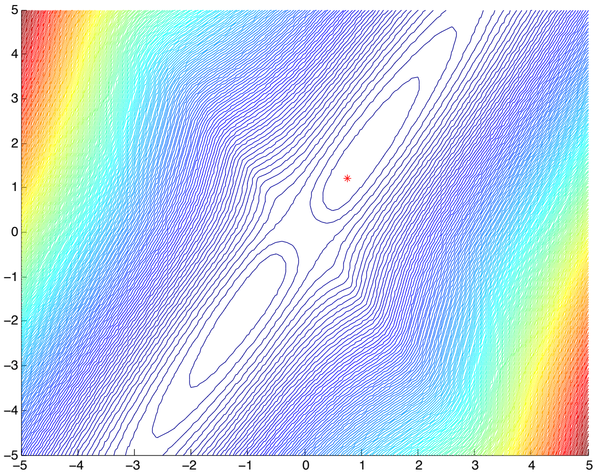
where $$\mathcal{A}(\omega_1,\omega_2) := A_0 + \omega_1^2 A_1 + \omega_2^2 A_2 + \omega_1\omega_2 A_3$$

$$A_0 = \begin{bmatrix} -2 & 1 & -3 \\ 1 & -2 & 1.2 \\ -3 & 1.2 & -2 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0.3 & 5 & 0 \\ 5 & 0.3 & 0 \\ 0 & 0 & -0.7 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1.3 & 0 & 0 \\ 0 & -0.3 & 2 \\ 0 & 2 & -0.3 \end{bmatrix} \quad A_3 = \begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

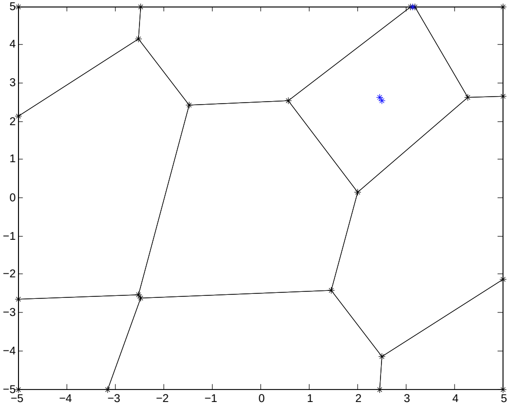
Remark: $\gamma = \lambda_{\min}\left(\begin{bmatrix} 2A_1 & A_3 \\ A_3 & 2A_2 \end{bmatrix}\right) \leq \lambda_{\min}\left[\nabla^2 \lambda_1\left(\mathcal{A}(\omega_1,\omega_2)\right)\right]$
for all ω_1, ω_2 such that $\lambda_1\left(\mathcal{A}(\omega_1,\omega_2)\right)$ is simple.

Minimizing Largest Eigenvalue



$$\omega_* = (0.7661, 1.2217)$$

Progress of the subregions



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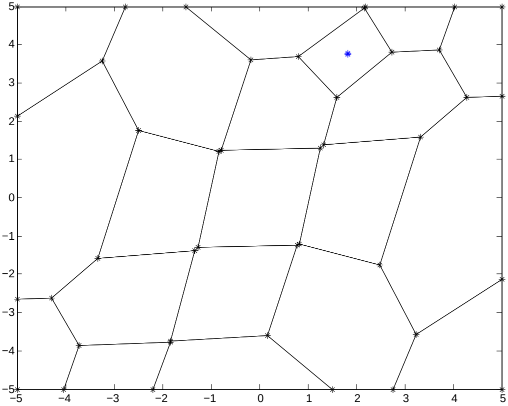

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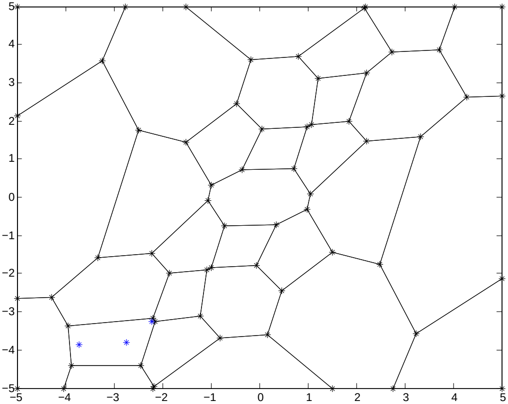

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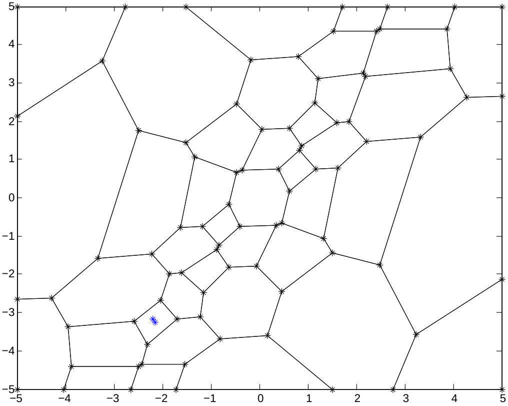

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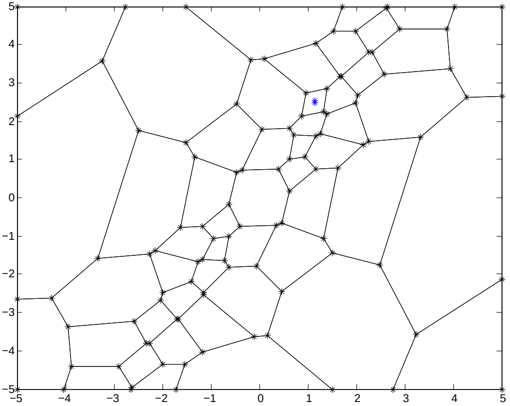

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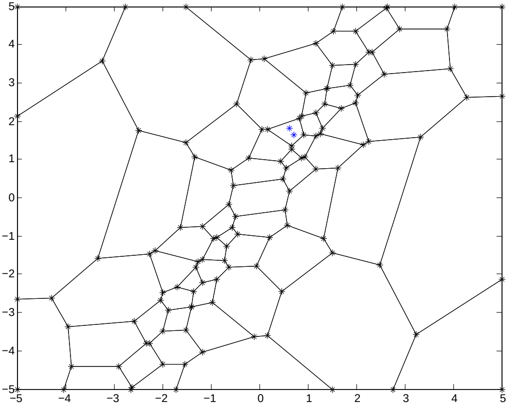

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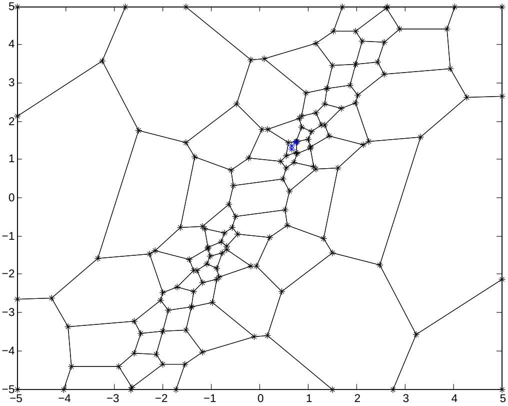

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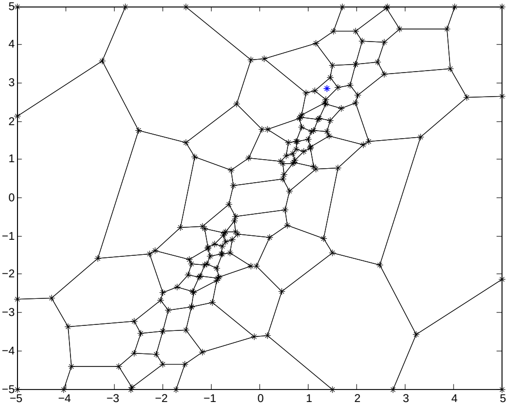

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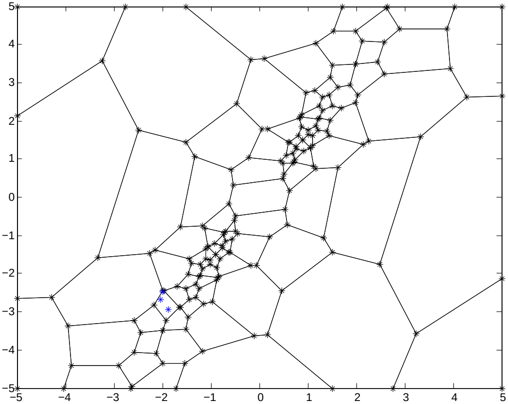

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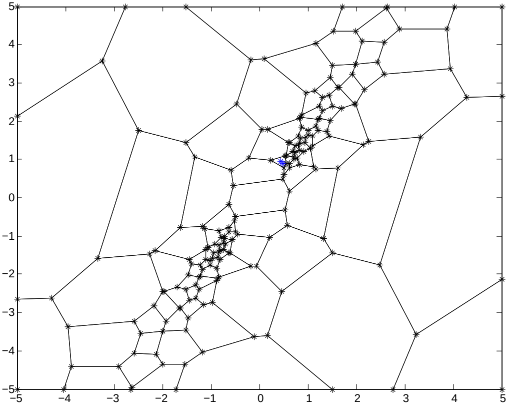

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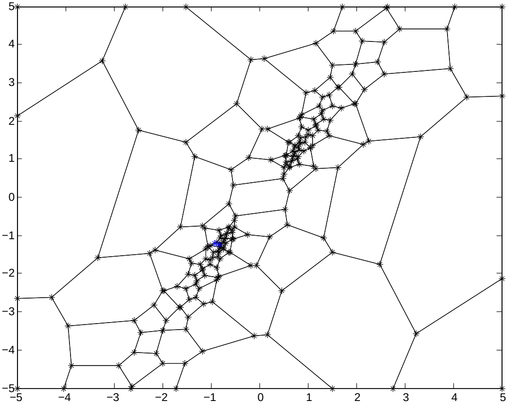

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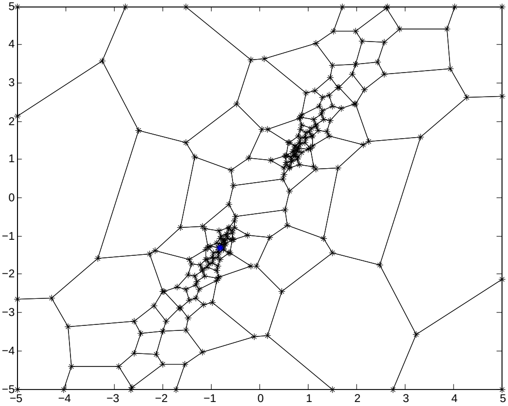


Nearest Polynomial with a Multiple Eigenvalue

The distance from a 5×5 random quadratic matrix polynomial $P(\omega)$ to a nearest one with a multiple eigenvalue

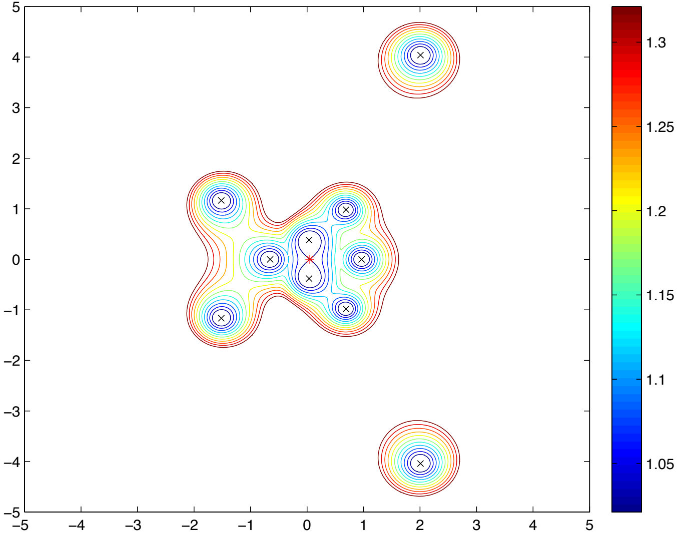
$$\tau_2 := \min_{\omega \in \mathbb{C}} \underbrace{\max_{\gamma \in \mathbb{R}} \sigma_{2n-1} \left(\begin{bmatrix} P(\omega) & \gamma P'(\omega) \\ 0 & P(\omega) \end{bmatrix} \right)}_{\lambda(\omega) :=} .$$

Connected to the ϵ-pseudospectrum of P

$$\Lambda_\epsilon(P) := \bigcup_{\|\Delta\|_2 \leq \epsilon} \Lambda\left(P(\omega) + \Delta\right).$$

Nearest Polynomial with a Multiple Eigenvalue

The inner-most curve is the boundary of the ϵ-pseudospectrum for $\epsilon = \tau_2 = 0.3211$ (computed by the algorithm).



Rate of Convergence

ϵ	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
eigopt	46	59	69	79	89	98
brute force	881	8812	88125	881249	8815191	86070462
direct	25	51	61	105	245	597

Number of function evaluations on a 1D (numerical radius) example with respect to absolute accuracy ϵ

ϵ	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
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Number of function evaluations on a 2D (distance to uncontrollability) example with respect to absolute accuracy ϵ

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Optimization with an Eigenvalue Constraint

Problem

Given a matrix-valued function $\mathcal{A}(\omega) : \mathbb{R}^d \to \mathbb{C}^{n \times n}$, that is

- analytic on $\mathbb{R}^d$, and
- such that $\mathcal{A}(\omega)^* = \mathcal{A}(\omega) \;\; \forall \omega \in \mathbb{R}^d$,

and a $c \in \mathbb{R}^d$.

Constrained Eigenvalue Optimization

$$\begin{array}{ll} \max_{\omega \in \mathbb{R}^d} & c^T \omega \\ \text{subject to} & \lambda_n \left(\mathcal{A}(\omega) \right) \leq 0 \end{array}$$

Applications

For a given $A \in \mathbb{R}^{n \times n}$,
$\alpha_\epsilon(A)$ - the real part of the rightmost point in $\Lambda_\epsilon(A)$
$\rho_\epsilon(A)$ - the modulus of the outermost point in $\Lambda_\epsilon(A)$

$$\Lambda_\epsilon(A) := \bigcup_{\|\Delta\|_2 \leq \epsilon} \Lambda(A + \Delta)$$
$$= \{z \in \mathbb{C} \mid \sigma_n(A - zI) \leq \epsilon\}$$

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$\alpha_\epsilon(A)$ - the real part of the rightmost point in $\Lambda_\epsilon(A)$
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$$\Lambda_\epsilon(A) := \bigcup_{\|\Delta\|_2 \leq \epsilon} \Lambda(A + \Delta)$$
$$= \{z \in \mathbb{C} \mid \sigma_n(A - zI) \leq \epsilon\}$$

Eigenvalue Optimization Characterization

$$\max_{\omega \in \mathbb{R}^d} \quad \omega_1$$
$$\text{subject to} \quad \sigma_n(\mathcal{P}(\omega)) \leq \epsilon$$

$\alpha_\epsilon(A): \ \mathcal{P}(\omega) = A - (\omega_1 + i\omega_2)I, \quad \rho_\epsilon(A): \ \mathcal{P}(\omega) = A - \omega_1 e^{i\omega_2} I$

Applications

For a given $A \in \mathbb{R}^{n \times n}$,
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Eigenvalue Optimization Characterization

$$\max_{\omega \in \mathbb{R}^d} \quad \omega_1$$

$$\text{subject to} \quad \lambda_n\left(\mathcal{A}(\omega)\right) \leq 0$$

$\alpha_\epsilon(A): \quad \mathcal{A}(\omega) = [A - (\omega_1 + i\omega_2)I]^* [A - (\omega_1 + i\omega_2)I] - \epsilon^2 I$
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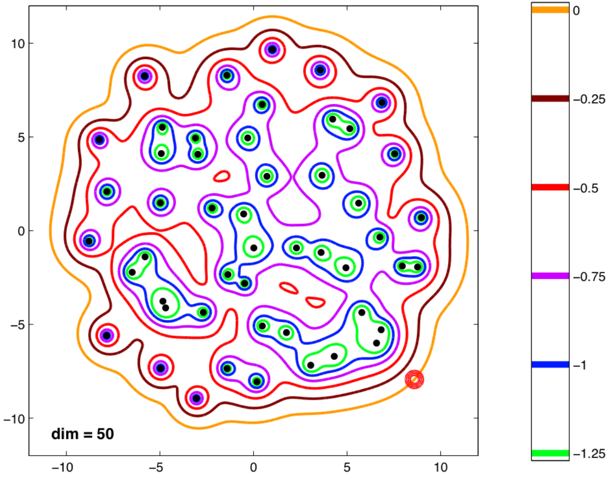
Applications

$\Lambda_\epsilon(A)$ for various ϵ and $\rho_\epsilon(A)$ (red circle) of a 50×50 matrix



Support Function Based Approach

Theorem (Quadratic Support Functions)

Suppose $\tilde{\omega}$ is such that $\lambda_n(\tilde{\omega}) := \lambda_n(\mathcal{A}(\tilde{\omega}))$ is simple, and γ satisfies

$$\lambda_{\max}\left[\nabla^2 \lambda_n(\omega)\right] \leq \gamma$$

for all $\omega \in \mathcal{B}_d$ such that $\lambda_n(\omega)$ is simple. Then

$$s(\omega; \tilde{\omega}) := \lambda_n + \nabla \lambda_n^T(\omega - \tilde{\omega}) + \frac{\gamma}{2}\|\omega - \tilde{\omega}\|^2$$

is an upper support function for $\lambda_n(\omega)$, where $\lambda_n := \lambda_n(\tilde{\omega})$ and $\nabla \lambda_n := \nabla \lambda_n(\tilde{\omega})$.

Support Functions for Extreme Eigenvalues

Theorem (Deducing γ analytically)

The eigenvalue function $\lambda_n(\omega) := \lambda_n(\mathcal{A}(\omega))$ satisfies

$$\lambda_{\max}\left[\nabla^2 \lambda_n(\omega)\right] \leq \lambda_{\max}\left[\nabla^2 \mathcal{A}(\omega)\right]$$

for each ω such that $\lambda_n(\omega)$ is simple.

$$\nabla^2 \mathcal{A}(\omega) := \begin{bmatrix} \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_1^2} & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_1 \partial \omega_2} & \cdots & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_1 \partial \omega_d} \\ \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_2 \partial \omega_1} & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_2^2} & \cdots & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_2 \partial \omega_d} \\ & & \ddots & \\ \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_d \partial \omega_1} & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_d \omega_2} & \cdots & \frac{\partial^2 \mathcal{A}(\omega)}{\partial \omega_d^2} \end{bmatrix}.$$

Support Function Based Approach

Convexify the constrained eigenvalue optimization problem

Convex Program

$$\max_{\omega \in \mathbb{R}^d} \quad c^T \omega$$

$$\text{subject to} \quad \lambda_n(\mathcal{A}(\omega_k)) + \nabla\lambda_n^T(\mathcal{A}(\omega_k))(\omega - \omega_k) + \frac{\gamma}{2}\|\omega - \omega_k\|^2 \leq 0$$

Generate a sequence $\{\omega_k\}$ such that ω_{k+1} is the maximizer of the convex program.

$$\omega_{k+1} = \omega_k + \frac{1}{\gamma}\left[\frac{1}{\mu_+} \cdot c - \nabla\lambda_k\right], \quad \text{where} \quad \mu_+ = \frac{\|c\|}{\sqrt{\|\nabla\lambda_k\|^2 - 2\gamma\lambda_k}}.$$

$\lambda_k := \lambda_n(\mathcal{A}(\omega_k))$ and $\nabla\lambda_k := \nabla\lambda_n(\mathcal{A}(\omega_k))$

Note: If ω_0 is feasible, ω_k is feasible $\forall k$.

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Theorem (M)

Suppose $\{\omega_k\}$ *is such that* $\lambda_n(\mathcal{A}(\omega_k))$ *is simple, and* $\nabla\lambda_n(\mathcal{A}(\omega_k)) \neq 0$ *for each* $k \in \mathbb{N}$*. Then*

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Suppose $\{\omega_k\}$ *is such that* $\lambda_n(\mathcal{A}(\omega_k))$ *is simple, and there exists a real scalar* $m > 0$ *satisfying* $\|\nabla\lambda_n(\mathcal{A}(\omega_k))\| \geq m$ *for each* $k \in \mathbb{N}$*. Then* $\lim_{k\to\infty} \theta_k = 0$ *where*

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Convergence

Two results imply convergence even in nonsmooth case:

The eigenvalue function $\lambda_n(\mathcal{A}(\omega))$ is differentiable everywhere except on a subset Ω of $\mathbb{R}^d$ of measure zero. In this case the generalized gradient is given by

$$\partial\lambda_n(\mathcal{A}(\omega)) := \mathrm{co}\left\{ \lim_{k\to\infty} \nabla\lambda_n(\mathcal{A}(\tilde{\omega}_k)) \mid \tilde{\omega}_k \to \omega, \;\; \tilde{\omega}_k \notin \Omega \, \forall k \right\}.$$

Letting $\omega_* := \lim_{k\to\infty} \omega_k$, from Theorem (convergence)

$$\tilde{\mu} c = \lim_{k\to\infty} \nabla\lambda_n(\mathcal{A}(\omega_k)) \in \partial\lambda_n(\mathcal{A}(\omega_*))$$

where $\tilde{\mu} = \|c\|/(\lim_{k\to\infty} \|\nabla\lambda_n(\mathcal{A}(\omega_k))\|)$. Thus ω_* satisfies the first order necessary conditions

$$\exists \mu > 0 \quad \text{s.t.} \quad c \in \mu \cdot \partial\lambda_n(\mathcal{A}(\omega_*)) \qquad \text{and} \qquad \lambda_n(\mathcal{A}(\omega_*)) = 0.$$

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Example

ϵ-pseudospectral abscissa and radius

$$\max_{\omega \in \mathbb{R}^d} \quad \omega_1$$

$$\text{subject to} \quad \lambda_n\left(\mathcal{A}(\omega)\right) \leq 0$$

$$\alpha_\epsilon(A): \quad \mathcal{A}(\omega) = [A - (\omega_1 + i\omega_2)I]^* \, [A - (\omega_1 + i\omega_2)I] - \epsilon^2 I$$

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ϵ-pseudospectral abscissa

$$\nabla^2 \mathcal{A}(\omega) = \begin{bmatrix} \frac{\partial \mathcal{A}^2(\omega)}{\partial \omega_1^2} & \frac{\partial \mathcal{A}^2(\omega)}{\partial \omega_1 \partial \omega_2} \\ \frac{\partial \mathcal{A}^2(\omega)}{\partial \omega_2 \partial \omega_1} & \frac{\partial \mathcal{A}^2(\omega)}{\partial \omega_2^2} \end{bmatrix} = 2I$$

$$\lambda_{\max}\left[\nabla^2 \lambda_n(\omega)\right] \leq \lambda_{\max}\left[\nabla^2 \mathcal{A}(\omega)\right] = \gamma := 2$$

ϵ-pseudospectral radius

$$\nabla^2 \mathcal{A}(\omega) = \begin{bmatrix} 2I & -2\Im(e^{-i\omega_2}A) \\ -2\Im(e^{-i\omega_2}A) & 2\Re(\omega_1 e^{-i\omega_2}A) \end{bmatrix}.$$

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$$\max\left(2 + 2\|A\|, 2\epsilon\|A\| + 2\|A\|^2 + 2\|A\|\right) \text{ (Gersgorin's theorem)}$$

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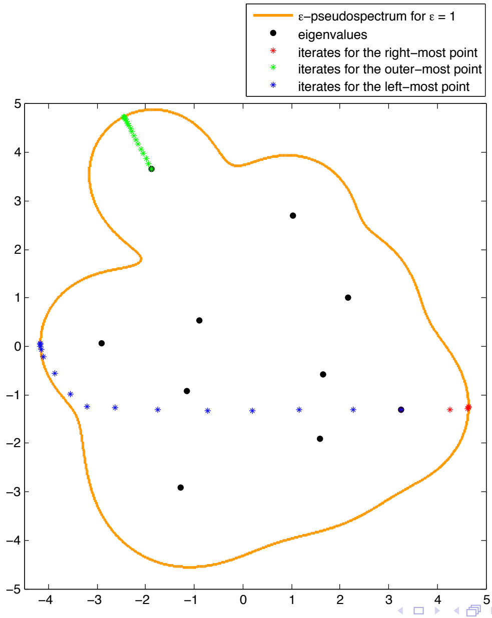
ε–pseudospectrum for ε = 1
eigenvalues
iterates for the right–most point
iterates for the outer–most point
iterates for the left–most point
5
4
3
2
1
0
–1
–2
–3
–4
–5
–4
–3
–2
–1
0
1
2
3
4
5

Future

Applications to engineering problems, such as those arising from structural design and control theory

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- Rate of convergence analysis
- Analysis of growth in the number of vertices; When is the algorithm computationally feasible?

Constrained Optimization

- Extensions for convex objectives with multiple eigenvalue constraints and quadratic constraints

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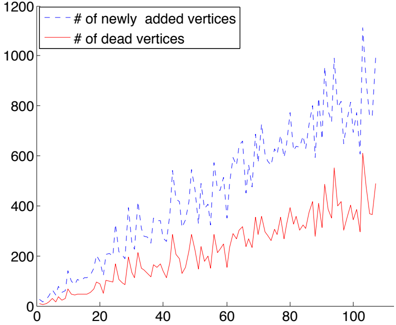
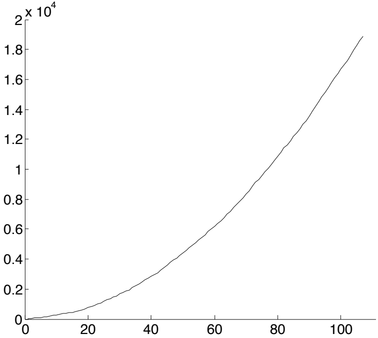
Growth in the number of vertices ($d = 5$)





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- E. Mengi. A support function based algorithm for optimization with eigenvalue constraints. *SIAM J Optim*, Submitted.

eigopt and **eigopt_constrained** (software) available at
http://home.ku.edu.tr/∼emengi/software/eigopt.html
http://home.ku.edu.tr/∼emengi/software/eigopt_constrained.tar

References

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