A POSTERIORI ERROR ESTIMATES FOR THE STOKES PROBLEM *

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Abstract. We derive and analyze an a posteriori error estimate for the mini-element discretization of the Stokes equations. The estimate is based on the solution of a local Stokes problem in each element of the finite element mesh, using spaces of quadratic bump functions for both velocity and pressure errors. This results in solving a 9×9 system which reduces to two 3×3 systems easily invertible. Comparisons with other estimates based on a Petrov-Galerkin solution are used in our analysis, which shows that it provides a reasonable approximation of the actual discretization error. Numerical experiments clearly show the efficiency of such an estimate in the solution of self adaptive mesh refinement procedures.

Key words. Mixed finite element methods, Stokes equations, a posteriori error estimates, mesh adaptation, mini-element formulation, Petrov-Galerkin formulation.

AMS subject classifications. 65F10, 65N20, 65N30.

1. Introduction. The need for accurate solutions of large scale problems (in particular) in Computational Fluid Dynamics has made the use of adaptive, automatic re-meshing very attractive for finite element computations of approximations to solution of partial differential equations [2]. A posteriori error estimates/estimators were introduced in order to provide an information about the local and global quality of the computed finite element solution. They allow the automatic determination of the zones in the mesh which require some refinement or unrefinement.

In this paper we present and analyze an error estimate for the Stokes problem [13], which plays a center role in the solution of more complicated problems arising in particular in Computational Fluid Dynamics [12] [18] [13] [15] [14].

In section 2, we introduce the equations and the notations used. These equations were solved using a two-level iterative scheme applied to a mini-element discretization of the corresponding variational formulation [6].

Numerous *a priori* and *a posteriori* error estimates for elliptic problems [3], for indefinite problems like the Stokes equations [16] or more general problems [2] have already been derived. In particular our new estimate can be viewed as a simplification of the one presented in [16]. Because a direct analysis of this estimate was quite difficult, we compared the mini-element formulation [1] [7] [8] with the method proposed by T.J.R.Hughes et al. [10] for solving the Stokes problem, and performed our analysis on the Petrov-Galerkin scheme.

Section 3 contains a presentation of the error estimate, which is based on the solution of a *local* Stokes problem in each element. This estimate is shown to be both a global upper and lower bound of the discretization error, by comparing it to an estimate derived from the Petrov-Galerkin approach.

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In section 4, we test this estimate on several classical problems and demonstrate its efficiency in grid adaptation.

2. The Stokes Equations. In this section we consider a mixed finite element approximation of the following Stokes equations [9]:

Find $\boldsymbol{u}$ (velocity field, 2 components) $\in (\mathcal{H}^1(\Omega))^2$ (the usual Sobolev space) and p (pressure field) $\in \mathcal{L}^2(\Omega)$ (the usual Lebesgue space) such that

$$\left\{\begin{array}{rcll} -\nu\Delta\boldsymbol{u}+\nabla p & = & \boldsymbol{f} & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} & = & 0 & \text{in } \Omega \\ \boldsymbol{u} & = & \boldsymbol{g} & \text{on } \partial\Omega \end{array}\right. \tag{2.1}$$

in a bounded domain $\Omega \subset \mathcal{R}^2$. The function $\boldsymbol{f}$ is a smooth function on $\mathcal{R}^2$, $\boldsymbol{g}$ is piecewise linear and satisfies the compatibility condition $\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}ds = 0$. Furthermore $\int_\Omega p\, d\Omega$ is assumed to be 0. The constant ν is a viscosity parameter.

We define the spaces

$$\mathcal{H}^1_g(\Omega) = \{\boldsymbol{u} \in (\mathcal{H}^1(\Omega))^2,\ \boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega\ \} \tag{2.2}$$

$$\mathcal{L}^2_0(\Omega) = \{p \in \mathcal{L}^2(\Omega),\ \int_\Omega p\, d\Omega\ = 0\} \tag{2.3}$$

$$H_g = \mathcal{H}^1_g(\Omega) \times \mathcal{L}^2_0(\Omega) \tag{2.4}$$

and the two bilinear forms

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu \int_\Omega \nabla\boldsymbol{u}\nabla\boldsymbol{v}d\Omega \qquad \boldsymbol{u},\boldsymbol{v} \in (\mathcal{H}^1(\Omega))^2 \tag{2.5}$$

$$b(\boldsymbol{u},p) = -\int_\Omega p\ \nabla.\boldsymbol{u}d\Omega \qquad \boldsymbol{u} \in (\mathcal{H}^1(\Omega))^2, p \in \mathcal{L}^2(\Omega) \tag{2.6}$$

$(\cdot,\cdot)$ will denote the $\mathcal{L}^2$ inner product associated with the norm $\|\cdot\|$. We define the energy norm $|||(\cdot,\cdot)|||$ by

$$|||(\boldsymbol{u},p)|||^2 = a(\boldsymbol{u},\boldsymbol{u}) + \frac{1}{\nu}\|p\|^2 \qquad \boldsymbol{u} \in \mathcal{H}^1, p \in \mathcal{L}^2 \tag{2.7}$$

Then a classical variational formulation of the system of equations (2.1) reads

Find $(\boldsymbol{u},p) \in H_g$ such that

$$\left\{\begin{array}{lcll} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) & = & (f,\boldsymbol{v}) & \boldsymbol{v} \in (\mathcal{H}^1_0(\Omega))^2 \\ b(\boldsymbol{u},q) & = & 0 & q \in \mathcal{L}^2_0(\Omega) \end{array}\right. \tag{2.8}$$

It satisfies the following "inf-sup" condition:

$$\inf_{\substack{p \in \mathcal{L}^2(\Omega) \\ p \neq 0}} \ \sup_{\substack{\boldsymbol{u} \in (\mathcal{H}^1(\Omega))^2 \\ \boldsymbol{u} \neq 0}} \frac{b(\boldsymbol{u},p)}{\|\nabla\boldsymbol{u}\|.\|p\|} \geq \mu_1 > 0 \tag{2.9}$$

which guarantees existence and uniqueness of a solution (u,p) of (2.8) (or (2.1)).

Let $\mathcal{T}$ be a triangulation of Ω such that any two triangles in $\mathcal{T}$ share at most a vertex or an edge. For $\tau \in \mathcal{T}$ let h_τ be the diameter of τ and E_τ the set of (three)

edges on $\partial\tau$. Let $h = \max_{\tau\in\mathcal{T}} h_\tau$. The set E contains all interior edges and for $e \in E$ we denote by h_e the length of e. We suppose also that the triangulation $\mathcal{T}$ satisfies a minimal angle condition, i.e. the smallest angle in triangle $\tau \in \mathcal{T}$ is bounded away from zero by some constant independent of h. This is equivalent to

$$h_\tau^{-1} \min_{e\in E_\tau} h_e \geq C_1 > 0 \quad \text{and} \quad h_\tau^{-1} \max_{e\in E_\tau} h_e \leq C_2 \qquad \tau \in \mathcal{T} \tag{2.10}$$

Furthermore, for $\Gamma = e$, E, or some subset of E, we define the inner product and associated norm

$$\langle u, v\rangle_\Gamma = \int_\Gamma uvds \ = \sum_{e\in\Gamma} \int_e uvds$$

$$\|u\|_\Gamma = (\langle u, u\rangle_\Gamma)^{1/2}$$

Let $\mathcal{C}^0$ be the space of continuous functions over $\mathcal{T}$. Let $\psi_i = \psi_i(\tau), i = 1, 3$ be the barycentric coordinates (linear nodal basis functions) in the triangle τ. In order to facilitate the introduction of local function spaces and inner products we will need in our analysis, we consider the following spaces of piecewise $\mathcal{H}^1$ functions

$$\mathcal{H}_\mathcal{T} = \prod_{\tau\in\mathcal{T}} \mathcal{H}^1(\tau) = \{u, u_{|\tau} \in \mathcal{H}^1(\tau), \tau \in \mathcal{T}\} \tag{2.11}$$

and the spaces

$$\mathcal{L} = \prod_{\tau\in\mathcal{T}} \mathcal{L}_\tau = \prod_{\tau\in\mathcal{T}} span\{\psi_i(\tau), 1 \leq i \leq 3, \tau \in \mathcal{T}\} \tag{2.12}$$

$$\mathcal{K} = \prod_{\tau\in\mathcal{T}} \mathcal{K}_\tau = \prod_{\tau\in\mathcal{T}} span\{\psi_i(\tau)\psi_j(\tau), 1 \leq i < j \leq 3, \tau \in \mathcal{T}\} \tag{2.13}$$

$$\mathcal{B} = \prod_{\tau\in\mathcal{T}} \mathcal{B}_\tau = \prod_{\tau\in\mathcal{T}} span\{\psi_1(\tau)\psi_2(\tau)\psi_3(\tau), \tau \in \mathcal{T}\} \tag{2.14}$$

$$\bar{\mathcal{X}} = \mathcal{H}_\mathcal{T} \cap \mathcal{L} \cap \mathcal{C}^0 \tag{2.15}$$

$$\mathcal{X} = (\bar{\mathcal{X}} \oplus \mathcal{B})^2 \tag{2.16}$$

$$\mathcal{Y} = \mathcal{L}_0^2 \cap \mathcal{L} \cap \mathcal{C}^0 \tag{2.17}$$

and set $\mathcal{Q}_\tau = \mathcal{L}_\tau \oplus \mathcal{K}_\tau$ for $\tau \in \mathcal{T}$ and $\mathcal{Q} = \prod_{\tau\in\mathcal{T}} \mathcal{Q}_\tau$. $\mathcal{L}$ is the space of piecewise linear functions and $\mathcal{Q}$ the space of piecewise quadratic functions on $\mathcal{T}$. The elements of $\mathcal{B}$ are referred to as *bubble functions*.

For $\boldsymbol{u}, \boldsymbol{v} \in (\mathcal{H}_\mathcal{T})^2$ and $p \in \mathcal{L}_0^2$ the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are interpreted as

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\tau\in\mathcal{T}} a(\boldsymbol{u}, \boldsymbol{v})_\tau = \nu \sum_{\tau\in\mathcal{T}} \int_\tau \nabla\boldsymbol{u}\nabla\boldsymbol{v}d\tau \tag{2.18}$$

$$b(\boldsymbol{u}, p) = \sum_{\tau\in\mathcal{T}} b(\boldsymbol{u}, p)_\tau = -\sum_{\tau\in\mathcal{T}} \int_\tau p\, \nabla\cdot\boldsymbol{u}d\tau \tag{2.19}$$

Note that since $\mathcal{X}$ and $\mathcal{Y}$ are contained in $\mathcal{L}^2$ the $\mathcal{L}^2$-inner product on $\mathcal{X}$ and $\mathcal{Y}$ is the usual $\mathcal{L}^2$-inner product.

Problem (2.8) can be solved using a good choice of spaces for $\boldsymbol{u}$ and p.

For example, the *mini-element* discretization [1] of the system (2.1) is given by:

Find $(\boldsymbol{u}_h, p_h) \in \mathcal{X}_g \times \mathcal{Y}$ such that

$$\left\{ \begin{array}{lcl} a(\boldsymbol{u}_h, \boldsymbol{v}) \;+\; b(\boldsymbol{v}, p_h) & = & (\boldsymbol{f}, \boldsymbol{v}) \\ b(\boldsymbol{u}_h, q) & = & 0 \end{array} \right. \tag{2.20}$$

for all $(\boldsymbol{v}, q) \in \mathcal{X}_0 \times \mathcal{Y}$.

This formulation also satisfies an inequality of the type (2.9) [9],

$$\inf_{\substack{(\boldsymbol{u},p) \in \mathcal{X} \times \mathcal{Y} \\ (\boldsymbol{u},p) \neq 0}} \; \sup_{\substack{(\boldsymbol{v},q) \in \mathcal{X} \times \mathcal{Y} \\ (\boldsymbol{v},q) \neq 0}} \frac{a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},q) + b(\boldsymbol{v},p)}{|||(\boldsymbol{u},p)|||.|||(\boldsymbol{v},q)|||} \geq \mu_1 \tag{2.21}$$

for a constant $\mu_1 > 0$ which is independent of the mesh size h (both constants μ_1 in (2.21) and (2.9) are denoted with the same symbol since no confusion can arise). This condition implies the unique solvability of the system (2.20).

The decomposition $\boldsymbol{u}_h = \boldsymbol{u}_{h,l} + \boldsymbol{u}_{h,b}$, with $\boldsymbol{u}_{h,l} \in (\mathcal{L} \cap \mathcal{C}^0)^2$ and $\boldsymbol{u}_{h,b} \in \mathcal{B}^2$, is unique. A static condensation of the bubble unknowns $\boldsymbol{u}_{h,b}$ yields, for $\boldsymbol{v} \in \bar{\mathcal{X}}$ and $q \in \bar{\mathcal{Y}}$:

$$\left\{ \begin{array}{lclcl} a(\boldsymbol{u}_{h,l}, \boldsymbol{v}) & + & b(\boldsymbol{v}, p_h) & = & (\boldsymbol{f}, \boldsymbol{v}) \\ b(\boldsymbol{u}_{h,l}, q) & - & \displaystyle\sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma_\tau \nu} (\nabla p_h, \nabla q)_\tau & = & \displaystyle -\sum_{\tau \in \mathcal{T}} \frac{1}{60\sigma_\tau \nu} (\boldsymbol{f}, \psi_b)_\tau \nabla q \end{array} \right. \tag{2.22}$$

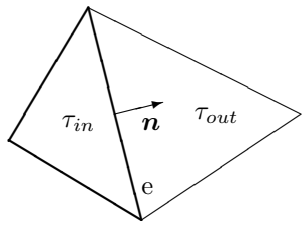
Here $\sigma_\tau = \dfrac{1}{|\tau|}(\nabla\psi_b(\tau), \nabla\psi_b(\tau))_\tau$ ($|\tau|$ represents the area of the triangle τ) and $\psi_b(\tau) = \psi_1(\tau)\psi_2(\tau)\psi_3(\tau)$ in triangle τ (bubble function). σ_τ is of order $\mathcal{O}(h_\tau^{-2})$ in the sense that there exist two positive constants C_3 and C_4 depending on the minimal angle in the triangulation such that

$$C_3\, h_\tau^2 \leq \sigma_\tau^{-1} \leq C_4\, h_\tau^2$$

Using elementwise integration by parts on (2.22), a simple calculation yields, for $(\boldsymbol{u}, p)$ solution of the system (2.1) and $(\boldsymbol{v}, p)$ in $\mathcal{H}_\mathcal{T} \times \mathcal{H}_\mathcal{T} \times \mathcal{L}_0^2$

$$\left\{ \begin{array}{rcl} a(\boldsymbol{u} - \boldsymbol{u_{h,l}}, \boldsymbol{v}) + b(\boldsymbol{v}, p - p_h) & = & (\boldsymbol{r}, \boldsymbol{v}) - \nu\langle \left[\dfrac{\partial(\boldsymbol{u} - \boldsymbol{u}_{h,l})}{\partial \boldsymbol{n}}\right]_A, [\boldsymbol{v}]_J \rangle \\ & & +\langle (p - p_h)\boldsymbol{n}, [\boldsymbol{v}]_J \rangle_E + \nu\langle \left[\dfrac{\partial \boldsymbol{u}_{h,l}}{\partial \boldsymbol{n}}\right]_J, [\boldsymbol{v}]_A \rangle_E \\ b(\boldsymbol{u} - \boldsymbol{u}_{h,l}, q) & = & (s, q) \end{array} \right. \tag{2.23}$$

where the vector $\boldsymbol{n} = (n_x, n_y)$ is the unit normal to a triangle, and average values ($[\cdot]_A$) and jump ($[\cdot]_J$) of a function v across an edge e are defined edgewise by:



$$[v]_A = \tfrac{1}{2}(v_{in} + v_{out})$$

and

$$[v]_J = v_{out} - v_{in}$$

Here τ_{in} denotes the current triangle. The quantities $\boldsymbol{r}$ and s are the residuals of (2.1) defined as

$$\left\{ \begin{array}{rcl} \boldsymbol{r} & = & \boldsymbol{f} - \nabla p_h \\ s & = & \nabla \cdot \boldsymbol{u}_{h,l} \end{array} \right. \tag{2.24}$$

The velocity term does not appear explicitly in the residual $\boldsymbol{r}$ since $\Delta \boldsymbol{u}_{h,l} = 0$. However, an integration by part of $(\boldsymbol{r}, \boldsymbol{v})$ yields :

$$\begin{aligned} (\boldsymbol{r}, \boldsymbol{v}) &= \sum_{\tau \in \mathcal{T}} (\boldsymbol{r}, \boldsymbol{v})_\tau \\ &= (\boldsymbol{f}, \boldsymbol{v}) - a(\boldsymbol{u}_{h,l}, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) - \langle \nu \left[\frac{\partial \boldsymbol{u}_{h,l}}{\partial \boldsymbol{n}} \right]_A - p_h \boldsymbol{n}, [\boldsymbol{v}]_J \rangle_E \\ &\quad - \nu \langle \left[\frac{\partial \boldsymbol{u}_{h,l}}{\partial \boldsymbol{n}} \right]_J, [\boldsymbol{v}]_A \rangle_E \end{aligned} \tag{2.25}$$

which is the computational form we will use in the following.

Note also that since the computed pressure is continuous, no jump of the quantity $p - p_h$ appears in (2.23).

A classical development of an a posteriori error estimate for this system is difficult, mainly because the mini-element discretization is not a member of a sequence of discretizations of varying degrees of approximation.

However, we can take advantage of the similarity between the mini-element formulation and the Petrov-Galerkin method of T. J. R. Hughes *et al* [10] using continuous piecewise linear interpolation for both pressure and velocity terms. Let $G = \mathcal{L} \cap \mathcal{C}^0$ or $\mathcal{Q} \cap \mathcal{C}^0$, and $(\boldsymbol{u_G}, p_G) \equiv (\boldsymbol{u_L}, p_L)$ when $G = \mathcal{L} \cap \mathcal{C}^0$ and $(\boldsymbol{u_G}, p_G) \equiv (\boldsymbol{u_Q}, p_Q)$ when $G = \mathcal{Q} \cap \mathcal{C}^0$. Then the Petrov-Galerkin formulation reads:

$$\left\{ \begin{array}{ll} a(\boldsymbol{u}_G, \boldsymbol{v}) + b(\boldsymbol{v}, p_G) & = (\boldsymbol{f}, \boldsymbol{v}) \quad \boldsymbol{v} \in G^2 \\ b(\boldsymbol{u}_G, q) - \displaystyle\sum_{\tau \in \mathcal{T}} \frac{1}{3600 \sigma_\tau \nu} ((\nabla p_G, \nabla q)_\tau - \nu(\Delta \boldsymbol{u}_G, \nabla q))_\tau & \\ & = - \displaystyle\sum_{\tau \in \mathcal{T}} \frac{1}{3600 \sigma_\tau \nu} (\boldsymbol{f}, \nabla q)_\tau \qquad q \in G \end{array} \right. \tag{2.26}$$

It is well known that the use of either piecewise linear or piecewise quadratic velocity and pressure terms yields a stable formulation, provided that the coefficient $(3600\sigma_\tau)^{-1}$ is small enough. We now show that the matrix of the resulting system is

non singular (equivalent to the fact that the formulation satisfies an inf-sup condition). To see this we will need the following lemma:

LEMMA 2.1.

$$\textit{For } \tau \in \mathcal{T} \qquad \|\Delta u\|_\tau^2 \le 2160\sigma_\tau \|\nabla u\|_\tau^2 \le 1944 \cdot 10^5 \sigma_\tau^2 \|u\|_\tau^2 \qquad u \in \mathcal{Q}_\tau$$

$$\textit{Also} \qquad \|\nabla(\nabla u)\|_\tau \le 2160\sqrt{15}\sigma_\tau \|u\|_\tau \qquad u \in \mathcal{K}_\tau$$

Proof. Let $\{1,2,3\}$ be a numbering of the vertices in the triangle τ and let θ_i be the angle at vertex $i, 1 \le i \le 3$. We have $\mathcal{Q}_\tau = span\{1, \psi_2, \psi_3, 4\psi_1\psi_2, 4\psi_2\psi_3, 4\psi_3\psi_1\}$. So let

$$u = u_1 + u_2\psi_2 + u_3\psi_3 + 4u_{1b}\psi_2\psi_3 + 4u_{2b}\psi_3\psi_1 + 4u_{3b}\psi_1\psi_2$$

and set $\boldsymbol{x}^t = (u_2, u_3, u_{1b}, u_{2b}, u_{3b})$. Define the 5×5 matrices

$$M = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

where the 2×2 matrix A, the 2×3 matrix B and the 3×3 matrices C and D are respectively defined by

$$A = \frac{1}{2}\begin{pmatrix} c_1 + c_3 & -c_1 \\ -c_1 & c_1 + c_2 \end{pmatrix} \quad B = -\frac{2}{3}\begin{pmatrix} -c_3 & c_1 + c_3 & -c_1 \\ -c_2 & -c_1 & c_1 + c_2 \end{pmatrix}$$

$$C = \frac{4}{3}\begin{pmatrix} c & -c_3 & -c_2 \\ -c_3 & c & -c_1 \\ -c_2 & -c_1 & c \end{pmatrix} \quad D = \frac{2880\sigma_\tau}{c}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}^t$$

Here $c_i = cotg(\theta_i)$ $(1 \le i \le 3)$ and $c = c_1 + c_2 + c_3$. Note also that $c = 180\sigma_\tau|\tau|$. Since

$$\|\Delta u\|_\tau^2 = \boldsymbol{x}^t M \boldsymbol{x} \ \text{ and } \ \|\nabla u\|_\tau^2 = \boldsymbol{x}^t N \boldsymbol{x}$$

we have the inequality

$$\|\Delta u\|_\tau^2 \le \lambda_1 \sigma_\tau \|\nabla u\|_\tau^2$$

with λ_1 being the largest (positive) root of $det(\frac{1}{\sigma_\tau}M - \lambda N) = 0$. The matrix M is positive semi definite and of rank 1 and N is positive definite, so that this equation has root 0 of multiplicity 4 and a positive root, which is λ_1. A direct calculation (add the last three rows in the determinant) shows that $\lambda_1 = 2160$, i.e. this value is independent of the geometry of the triangle τ.

We have also $\mathcal{Q}_\tau = span\{\psi_1, \psi_2, \psi_3, 4\psi_1\psi_2, 4\psi_2\psi_3, 4\psi_3\psi_1\}$. So let

$$u = u_1\psi_1 + u_2\psi_2 + u_3\psi_3 + 4u_{1b}\psi_2\psi_3 + 4u_{2b}\psi_3\psi_1 + 4u_{3b}\psi_1\psi_2$$

and set $\boldsymbol{y}^t = (u_1, u_2, u_3, u_{1b}, u_{2b}, u_{3b})$. Now we have

$$\|u\|_\tau^2 = \boldsymbol{y}^t S \boldsymbol{y} \ \text{ and } \ \|\nabla u\|_\tau^2 = \boldsymbol{y}^t N' \boldsymbol{y}$$

where

$$S = \frac{|\tau|}{180}\begin{pmatrix} 30 & 15 & 15 & 12 & 24 & 24 \\ 15 & 30 & 15 & 24 & 12 & 24 \\ 15 & 15 & 30 & 24 & 24 & 12 \\ 12 & 24 & 24 & 32 & 16 & 16 \\ 24 & 12 & 24 & 16 & 32 & 16 \\ 24 & 24 & 12 & 16 & 16 & 32 \end{pmatrix} \qquad N' = \begin{pmatrix} \alpha & \boldsymbol{z}^t \\ \boldsymbol{z} & N \end{pmatrix}$$

and $\alpha = \dfrac{c_2 + c_3}{2}$, $\boldsymbol{z}^t = \dfrac{1}{6}(-3c_3, -3c_2, -4(c_2 + c_3), 4c_3, 4c_2)$. The matrix S is positive definite while N' is now positive semi-definite, which implies that the maximal eigenvalue of N' is smaller than the trace $Tr(N')$ of N'. Thus

$$\boldsymbol{y}^t N' \boldsymbol{y} \leq Tr(N')\boldsymbol{y}^t\boldsymbol{y} = 5c\ \boldsymbol{y}^t\boldsymbol{y}$$

Since the minimal eigenvalue of S is $\lambda = \dfrac{31 - \sqrt{901}}{90}|\tau| \simeq 1.093 \cdot 10^{-2}|\tau| > 10^{-2}|\tau|$, we get:

$$\boldsymbol{y}^t N' \boldsymbol{y} \leq \frac{500c}{|\tau|}\boldsymbol{y}^t S \boldsymbol{y} = 9 \cdot 10^4 \sigma_\tau \boldsymbol{y}^t S \boldsymbol{y}$$

which was to be shown.

Finally, for $u \in \mathcal{K}_\tau$, let $u = 4u_{1b}\psi_2\psi_3 + 4u_{2b}\psi_3\psi_1 + 4u_{3b}\psi_1\psi_2$. Then, in the two-dimensional plane (x, y):

$$\begin{aligned}
\|\nabla(\nabla u)\|_\tau^2 &= |\tau|\ (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) \\
&= 32|\tau| \Bigg(2(\sum_{i=1}^{3} u_{ib}\psi_{i+1,x}\psi_{i+2,x})^2 + 2(\sum_{i=1}^{3} u_{ib}\psi_{i+1,y}\psi_{i+2,y})^2 \\
&\quad + (\sum_{i=1}^{3} u_{ib}(\psi_{i+1,x}\psi_{i+2,y} + \psi_{i+1,y}\psi_{i+2,x}))^2 \Bigg) \\
&\leq 192|\tau|\sum_{i=1}^{3} u_{ib}^2 \|\nabla\psi_{i+1}\|_\tau^2\ \|\nabla\psi_{i+2}\|_\tau^2 \\
&\leq 768 \cdot 90^2 |\tau| \sigma_\tau^2 \sum_{i=1}^{3} u_{ib}^2 \\
&\leq 192 \cdot 90^2 \cdot 45 \sigma_\tau^2 \|u\|_\tau^2
\end{aligned}$$

where the indices x, y, xx, xy and yy denote partial derivatives, and the indices $i+1$ and $i+2$ represent the modulo 3 values of $i+1$ and $i+2$ in the set $\{1, 2, 3\}$. □

These inequalities also apply to $\boldsymbol{u} \in \mathcal{Q}_\tau^k$ (or $\boldsymbol{u} \in \mathcal{K}_\tau^k$), $k > 1$, by simply adding the results on each component.

Let

$$\mathcal{S} = \{(\boldsymbol{u}, p) \in (\mathcal{Q} \cap \mathcal{C}^0)^3, \boldsymbol{u} = \boldsymbol{g} \ \text{ on } \ \partial\Omega, \int_\Omega p d\Omega \ = 0\}$$

Introducing the (nonsymmetric) bilinear form

$$B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) + \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma_\tau \nu}\left((\nabla p, \nabla q)_\tau - \nu(\Delta \boldsymbol{u}, \nabla q)_\tau\right)$$

and the semi-norm $\|\cdot\|_*$ defined by

$$\|(\boldsymbol{u},p)\|_*^2 \equiv \nu\|\nabla\boldsymbol{u}\|^2 + \sum_{\tau\in\mathcal{T}} \frac{1}{3600\sigma_\tau\nu}\|\nabla p\|_\tau^2$$

we have then

THEOREM 2.2. *For* $(\boldsymbol{u},p) \in \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}$ *we have*

$$B((\boldsymbol{u},p),(\boldsymbol{u},p)) \geq \omega\|(\boldsymbol{u},p)\|_*^2$$

Proof. It is similar to the proof of the coercivity of the very same bilinear form in [8] for a special choice of the stability constant $(3600\sigma_\tau)^{-1}$ for $\tau\in\mathcal{T}$. A direct application of Lemma 2.1 yields

$$\begin{aligned}
B((\boldsymbol{u},p),(\boldsymbol{u},p)) &= a(\boldsymbol{u},\boldsymbol{u}) + \sum_{\tau\in\mathcal{T}} \frac{1}{3600\sigma_\tau\nu}\left(\|\nabla p\|_\tau^2 - \nu(\Delta\boldsymbol{u},\nabla p)_\tau\right)\\
&\geq a(\boldsymbol{u},\boldsymbol{u}) + \sum_{\tau\in\mathcal{T}} \frac{1}{3600\sigma_\tau\nu}\left(\|\nabla p\|_\tau^2 - \nu\sqrt{2160\sigma_\tau}\|\nabla\boldsymbol{u}\|_\tau\|\nabla p\|_\tau\right)\\
&\geq (1-\frac{\sqrt{3}}{2\sqrt{5}})\left(\nu\|\nabla\boldsymbol{u}\|^2 + \sum_{\tau\in\mathcal{T}} \frac{1}{3600\sigma_\tau\nu}\|\nabla p\|_\tau^2\right)\\
&= (1-\frac{\sqrt{3}}{2\sqrt{5}})\|(\boldsymbol{u},p)\|_*^2 \qquad (2.27)
\end{aligned}$$

and the theorem holds with $\omega = 1-0.5\sqrt{0.6}\simeq 0.612$. Then, since the semi-norm is a norm on subspaces of $\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}$ (like $\mathcal{K}\times\mathcal{K}\times\mathcal{K}$) the form $B(\cdot,\cdot)$ is positive definite on these same subspaces (and in particular on $\mathcal{K}\times\mathcal{K}\times\mathcal{K}$). ∎

Note that $|||\cdot|||$ and $\|\cdot\|_*$ define two norms on $\mathcal{K}\times\mathcal{K}\times\mathcal{K}$ which are equivalent, i.e.

$$\beta_1\|(\boldsymbol{u},p)\|_* \leq |||(\boldsymbol{u},p)||| \leq \beta_2\|(\boldsymbol{u},p)\|_* \qquad (\boldsymbol{u},p)\in\mathcal{K}\times\mathcal{K}\times\mathcal{K}$$

for some positive constants β_1 and β_2. The first inequality results from a direct application of Lemma 2.1, and is more generally valid on $\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}$ ($\beta_1 = 25$), while the second one is a consequence of the minimal angle condition expressed by the inequality

$$\|\chi\|_\tau \leq \frac{c_5}{\sqrt{\sigma_\tau}}\|\nabla\chi\|_\tau \quad \text{for } \chi\in\mathcal{K}_\tau \qquad (2.28)$$

It is true in particular in $\mathcal{K}\times\mathcal{K}\times\mathcal{K}$ (but not all of $\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}$). As an indication, $c_5 < 2$ when the minimal angle in the triangulation is greater than 1 degree, which gives $\beta_2 < 120$ ($c_5 < 0.4$ for a minimal angle greater than 5 degrees, or $\beta_2 < 24$).

Another important case where the semi-norms are equivalent is the following

LEMMA 2.3. *Let* $(\boldsymbol{u},p)\in\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}$ *such that*

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = 0 \qquad \textit{all } \boldsymbol{v}\in\mathcal{L}\times\mathcal{L}$$

Then there exist two positive constants β_3 and β_4 such that

$$\beta_3 \|(\boldsymbol{u}, p)\|_* \leq |||(\boldsymbol{u}, p)||| \leq \beta_4 \|(\boldsymbol{u}, p)\|_*$$

Proof. Our proof is based on the duality argument given in [7].

Consider the dual problem

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \rho &= 0 \quad \text{in } \Omega \\ \nabla \cdot \boldsymbol{w} &= p \quad \text{in } \Omega \\ \boldsymbol{w} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

so that the regularity inequality

$$\|\nabla \boldsymbol{w}\| + \|\rho\| \leq c_6 \|p\|$$

holds. We have then

$$\|p\|^2 = (p, \nabla \cdot \boldsymbol{w}) = (p, \nabla \cdot \boldsymbol{v}) + (p, \nabla \cdot (\boldsymbol{w} - \boldsymbol{v}))$$

where $\boldsymbol{v}$ is the linear interpolator of $\boldsymbol{w}$ at the vertices of the triangulation so that

$$\|\boldsymbol{w} - \boldsymbol{v}\|_\tau \leq \frac{c_7}{\sqrt{\sigma_\tau}} \|\nabla \boldsymbol{w}\|_\tau$$

and

$$\|\nabla \boldsymbol{v}\|_\tau \leq \|\nabla \boldsymbol{w}\|_\tau + \|\nabla(\boldsymbol{v} - \boldsymbol{w})\|_\tau \leq c_8 \|\nabla \boldsymbol{w}\|_\tau$$

for all triangles τ ($\sigma_\tau = \mathcal{O}(h_\tau^{-2})$).

Integrating the second term by parts and using the hypothesis on $(\boldsymbol{u}, p)$ for the first term, we get

$$\begin{aligned} \|p\|^2 &= a(\boldsymbol{u}, \boldsymbol{v}) - (\nabla p, \boldsymbol{w} - \boldsymbol{v}) \\ &\leq \nu \|\nabla \boldsymbol{u}\| \, \|\nabla \boldsymbol{v}\| + \sum_{\tau \in \mathcal{T}} \|\nabla p\|_\tau \, \|\boldsymbol{w} - \boldsymbol{v}\|_\tau \\ &\leq \nu c_8 \|\nabla \boldsymbol{u}\| \, \|\nabla \boldsymbol{w}\| + c_7 \sum_{\tau \in \mathcal{T}} \frac{1}{\sqrt{\sigma_\tau}} \|\nabla p\|_\tau \, \|\nabla \boldsymbol{w}\|_\tau \\ &\leq \nu c_8 \|\nabla \boldsymbol{u}\| \, \|\nabla \boldsymbol{w}\| + c_7 \|\nabla \boldsymbol{w}\| \left(\sum_{\tau \in \mathcal{T}} \frac{\|\nabla p\|_\tau^2}{\sigma_\tau} \right)^{1/2} \\ &\leq \nu c_6 c_8 \|\nabla \boldsymbol{u}\| \, \|p\| + c_6 c_7 \|p\| \left(\sum_{\tau \in \mathcal{T}} \frac{\|\nabla p\|_\tau^2}{\sigma_\tau} \right)^{1/2} \end{aligned}$$

so that

$$\|p\| \leq \nu c_6 c_8 \|\nabla \boldsymbol{u}\| + c_6 c_7 \left(\sum_{\tau \in \mathcal{T}} \frac{\|\nabla p\|_\tau^2}{\sigma_\tau} \right)^{1/2}$$

Consequently

$$\nu^{-1}\|p\|^2 \leq 2c_6^2c_8^2\nu\|\nabla \boldsymbol{u}\|^2 + 2c_6^2c_7^2\sum_{\tau\in\mathcal{T}}\frac{\|\nabla p\|_\tau^2}{\nu\sigma_\tau}$$

and finally

$$|||(\boldsymbol{u},p)||| \leq \beta_4\|(\boldsymbol{u},p)\|_*$$

Conversely

$$\sum_{\tau\in\mathcal{T}}\frac{1}{3600\sigma_\tau}\|\nabla p\|_\tau^2 \leq 25\sum_{\tau\in\mathcal{T}}\|p\|_\tau^2 = 25\|p\|^2$$

which implies

$$\begin{aligned}\|(\boldsymbol{u},p)\|_*^2 &= \nu\|\nabla\boldsymbol{u}\|^2 + \sum_{\tau\in\mathcal{T}}\frac{1}{3600\sigma_\tau\nu}\|\nabla p\|_\tau^2\\ &\leq \nu\|\nabla\boldsymbol{u}\|^2 + 25\frac{\|p\|^2}{\nu}\\ &\leq 25|||(\boldsymbol{u},p)|||^2\end{aligned}$$

Note that the second part of the proof remains valid for any $(\boldsymbol{u},p)\in\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}$. □

In order to show the next result (continuity property) on the bilinear form B, we shall use the inequality

$$\|\nabla\cdot\boldsymbol{u}\|_\tau^2 \leq 2\,\|\nabla\boldsymbol{u}\|_\tau^2 \qquad \boldsymbol{u}\in\mathcal{H}_\mathcal{T}\times\mathcal{H}_\mathcal{T}$$

Lemma 2.4.

$$|B((\boldsymbol{u},p),(\boldsymbol{v},q))| \leq 27|||(\boldsymbol{u},p)|||\;|||(\boldsymbol{v},q)||| \tag{2.29}$$

for all $(\boldsymbol{u},p)$ *and* $(\boldsymbol{v},q)$ *in* $\mathcal{Q}^3$.

Proof. A straightforward calculation leads to

$$\begin{aligned}|B((\boldsymbol{u},p),(\boldsymbol{v},q))| &\leq \nu\|\nabla\boldsymbol{u}\|\;\|\nabla\boldsymbol{v}\| + \|p\|\;\|\nabla\cdot\boldsymbol{v}\| + \|q\|\;\|\nabla\cdot\boldsymbol{u}\|\\ &\quad+\sum_{\tau\in\mathcal{T}}\frac{\|\nabla p\|_\tau\|\nabla q\|_\tau}{3600\sigma_\tau\nu} + \sum_{\tau\in\mathcal{T}}\frac{\|\Delta\boldsymbol{u}\|_\tau\|\nabla q\|_\tau}{3600\sigma_\tau}\\ &\leq \nu\|\nabla\boldsymbol{u}\|\;\|\nabla\boldsymbol{v}\| + \sqrt{2}\|\nabla\boldsymbol{u}\|\;\|q\| + \sqrt{2}\|\nabla\boldsymbol{v}\|\;\|p\|\\ &\quad+\frac{25}{\nu}\sum_{\tau\in\mathcal{T}}\|p\|_\tau\;\|q\|_\tau + \sqrt{15}\sum_{\tau\in\mathcal{T}}\|\nabla\boldsymbol{u}\|_\tau\;\|q\|_\tau\\ &\leq \nu\|\nabla\boldsymbol{u}\|\;\|\nabla\boldsymbol{v}\| + \sqrt{2}\|\nabla\boldsymbol{u}\|\;\|q\| + \sqrt{2}\|\nabla\boldsymbol{v}\|\;\|p\|\\ &\quad+\frac{25}{\nu}\|p\|\;\|q\| + \sqrt{15}\|\nabla\boldsymbol{u}\|\;\|q\|\\ &\leq \left(18\nu\|\nabla\boldsymbol{u}\|^2 + 27\nu^{-1}\|p\|_\tau^2\right)^{1/2}\left(2\nu\|\nabla\boldsymbol{v}\|^2 + 27\nu^{-1}\|q\|^2\right)^{1/2}\\ &\leq 27|||(\boldsymbol{u},p)|||\;|||(\boldsymbol{v},q)|||\end{aligned}$$

□

The mini-element form after static condensation (2.22) and the Petrov-Galerkin form using piecewise linear velocities and pressure ($G = \mathcal{L}$ in (2.26)) are nearly identical: the stiffness matrices are identical and the right-hand sides differ by only a small quantity since

$$60(1, \psi_b)_\tau = |\tau|$$

so that

$$\begin{aligned} |(\boldsymbol{f}, \nabla q)_\tau - 60(\boldsymbol{f}, \psi_b)_\tau \nabla q| &= \left|(\boldsymbol{f} - \bar{\boldsymbol{f}}, (1 - 60\psi_b)\nabla q)_\tau\right| \\ &\leq c_9 h_\tau \|\nabla \boldsymbol{f}\|_\tau \|\nabla q\|_\tau \end{aligned}$$

where $\bar{\boldsymbol{f}}$ is the mean value of $\boldsymbol{f}$ in triangle τ and c_9 is a constant independent of τ. Consequently, we have

THEOREM 2.5.

$$\||(\boldsymbol{u}_L, p_L) - (\boldsymbol{u}_{h,l}, p_h)\|| \leq \frac{c_{10} h^2}{\sqrt{\nu}} \|\nabla \boldsymbol{f}\|$$

Proof. By Theorem 2.2 we have

$$\begin{aligned} \omega \|(\boldsymbol{u}_L, p_L) - (\boldsymbol{u}_{h,l}, p_h)\|_*^2 &\leq B((\boldsymbol{u}_L, p_L) - (\boldsymbol{u}_{h,l}, p_h), (\boldsymbol{u}_L, p_L) - (\boldsymbol{u}_{h,l}, p_h)) \\ &= \sum_{\tau \in \mathcal{T}} \frac{((\boldsymbol{f}, \nabla(p_L - p_h))_\tau - 60(\boldsymbol{f}, \psi_b \nabla(p_L - p_h))_\tau)}{3600 \sigma_\tau \nu} \\ &\leq \frac{c_9}{3600} \sum_{\tau \in \mathcal{T}} \frac{h_\tau}{\sigma_\tau \nu} \|\nabla \boldsymbol{f}\|_\tau . \|\nabla(p_L - p_h)\|_\tau \\ &\leq \frac{c_9 \sqrt{C_4} h^2}{\sqrt{3600\ \nu}} \sum_{\tau \in \mathcal{T}} \frac{1}{\sqrt{3600 \sigma_\tau \nu}} \|\nabla \boldsymbol{f}\|_\tau . \|\nabla(p_L - p_h)\|_\tau \\ &\leq \frac{c_9 \sqrt{C_4} h^2}{\sqrt{3600\ \nu}} \|\nabla \boldsymbol{f}\| \cdot \|(\boldsymbol{u}_L, p_L) - (\boldsymbol{u}_h, p_h)\|_* \end{aligned}$$

□

The linear part $(\boldsymbol{u}_{h,l}, p_h)$ of the computed solution $(\boldsymbol{u}_h, p_h)$ in the mini-element formulation is certainly a good approximation of the solution $(\boldsymbol{u}, p)$ of the initial problem, and that the bubble part of this solution is only introduced for stability reasons and does not improve the approximation of the velocity and pressure terms, as was pointed out in [16].

In the next section we take advantage of this special similarity between these two formulations in order to carry out the analysis of our error estimate.

3. An a posteriori error estimate based on the solution of a local Stokes problem. The goal of this section is to define an estimation of the discretization errors $(\boldsymbol{e}, \epsilon) = (\boldsymbol{u}, p) - (\boldsymbol{u_L}, p_L)$ and $(\boldsymbol{e}', \epsilon') = (\boldsymbol{u}, p) - (\boldsymbol{u_{h,l}}, p_h)$ for the Petrov-Galerkin and Mini-Element formulations respectively. The estimates are based on the solution of *local* Stokes problems (i.e. defined in each element). The analysis and derivation of these small problems extend the work done by Bank and Weiser for elliptic problems [5].

The main results presented here are Theorem 3.6 and Corollary 3.11, which show that the estimates provide a good global assessment of the discretization error, under reasonable assumptions.

Let $(\boldsymbol{e_Q}, \epsilon_Q) = (\boldsymbol{u_Q}, p_Q) - (\boldsymbol{u_L}, p_L)$. An error analysis on the system (2.26) similar to the one on (2.20) leads to the following system of equations:

$$
(3.1) \quad \begin{cases} a(\boldsymbol{e}, \boldsymbol{v}) + b(\boldsymbol{v}, \epsilon) &= (\boldsymbol{r}, \boldsymbol{v}) - \nu \langle \left[\dfrac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}}\right]_A, [\boldsymbol{v}]_J \rangle \\ & \quad + \langle \epsilon \boldsymbol{n}, [\boldsymbol{v}]_J \rangle + \nu \langle \left[\dfrac{\partial \boldsymbol{u_L}}{\partial \boldsymbol{n}}\right]_J, [\boldsymbol{v}]_A \rangle \\ b(\boldsymbol{e}, q) - \displaystyle\sum_{\tau \in \mathcal{T}} \frac{1}{3600 \sigma_\tau \nu} \Big((\nabla \epsilon, \nabla q)_\tau - \nu (\Delta \boldsymbol{e}, \nabla q)_\tau \Big) &= (s, q) \\ & \quad - \displaystyle\sum_{\tau \in \mathcal{T}} \frac{1}{3600 \sigma_\tau \nu} (\boldsymbol{r}, \nabla q)_\tau \end{cases}
$$

Equations (3.1) are valid for all $(\boldsymbol{v}, q) \in (H(\mathcal{T}))^3$.

This is equivalent to

$$
(3.2) \quad \begin{cases} B((\boldsymbol{e}, \epsilon), (\boldsymbol{v}, q)) &\equiv F((\boldsymbol{v}, q)) \\ & = L((\boldsymbol{v}, q)) + J((\boldsymbol{e}, \epsilon), (\boldsymbol{v}, q)) \qquad (\boldsymbol{v}, q) \in (H(\mathcal{T}))^3. \end{cases}
$$

with the forms $L(\cdot)$ and $J(\cdot)$ defined by

$$
L((\boldsymbol{v}, q)) = (\boldsymbol{r}, \boldsymbol{v}) - (s, q) + \nu \langle \left[\frac{\partial \boldsymbol{u_L}}{\partial \boldsymbol{n}}\right]_J, [\boldsymbol{v}]_A \rangle + \sum_{\tau \in \mathcal{T}} \frac{1}{3600 \sigma_\tau \nu} (\boldsymbol{r}, \nabla q)_\tau
$$

$$
J((\boldsymbol{e}, \epsilon), (\boldsymbol{v}, q)) = -\nu \langle \left[\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}}\right]_A, [\boldsymbol{v}]_J \rangle + \langle \epsilon \boldsymbol{n}, [\boldsymbol{v}]_J \rangle
$$

and the residuals $\boldsymbol{r}$ and s have the same meaning as in (2.24) with $(\boldsymbol{u}_{h,l}, p_h)$ replaced by $(\boldsymbol{u_L}, p_L)$.

Note that $J((\boldsymbol{v}, q)) = 0$ when $\boldsymbol{v}$ is continuous. From equations (3.2) we have for $(\boldsymbol{v}, q) \in (\mathcal{L} \cap \mathcal{C}^0)^3$

$$
(3.3) \qquad B((\boldsymbol{e_Q}, \epsilon_Q), (\boldsymbol{v}, q)) = B((\boldsymbol{u_Q}, p_Q), (\boldsymbol{v}, q)) - B((\boldsymbol{u_L}, p_L), (\boldsymbol{v}, q)) \ = \ 0
$$

and in particular

$$
a(\boldsymbol{e}_Q, \boldsymbol{v}) + b(\boldsymbol{v}, \epsilon_Q) = 0
$$

which gives, using Lemma 2.3,

$$
\beta_3 \|(\boldsymbol{e}_Q, \epsilon_Q)\|_* \leq |||(\boldsymbol{e}_Q, \epsilon_Q)||| \leq \beta_4 \|(\boldsymbol{e}_Q, \epsilon_Q)\|_*
$$

Note that the classical relation

$$
(3.4) \qquad B((\boldsymbol{e}, \epsilon), (\boldsymbol{v}, q)) = B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) - B((\boldsymbol{u_L}, p_L), (\boldsymbol{v}, q)) \ = \ 0
$$

also holds.

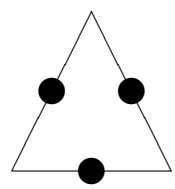
For $(\boldsymbol{v}, q) \in (\mathcal{Q} \cap \mathcal{C}^0)^3$ we also have

$$
\begin{aligned}
B((\boldsymbol{e_Q}, \epsilon_Q), (\boldsymbol{v}, q)) &= B((\boldsymbol{u_Q}, p_Q), (\boldsymbol{v}, q)) - B((\boldsymbol{u_L}, p_L), (\boldsymbol{v}, q)) \\
&= B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) - B((\boldsymbol{u_L}, p_L), (\boldsymbol{v}, q)) \\
&= B((\boldsymbol{e}, \epsilon), (\boldsymbol{v}, q)) \\
&= L((\boldsymbol{v}, q))
\end{aligned} \tag{3.5}
$$

We now define the *local* error estimate $(\check{\boldsymbol{e}}, \check{\epsilon}) \in \mathcal{K}^3$ by

$$
B((\check{\boldsymbol{e}}, \check{\epsilon}), (\boldsymbol{v}, q))_\tau = L((\boldsymbol{v}, q))_\tau \qquad (\boldsymbol{v}, q) \in \mathcal{K}^3(\tau) \tag{3.6}
$$

Since the bilinear $B(\cdot, \cdot)$ form is positive definite on $\mathcal{K}^3$, this estimate is well defined, on each triangle $\tau \in \mathcal{T}$. The velocity components as well as the pressure component are described in each triangle τ by quadratic bump functions, which schematically correspond to the following degrees of freedom:



degrees of freedom for velocity and pressure errors

Equation (3.6) is a 9×9 system to be solved in each triangle. Note that using an integration by parts on $(\boldsymbol{r}, \boldsymbol{v})_\tau$ similar to the one performed to get (2.25) the right-hand side $L((\boldsymbol{v}, q))_\tau$ takes the form

$$
\begin{aligned}
L((\boldsymbol{v}, q))_\tau = (\boldsymbol{f}, \boldsymbol{v})_\tau - a(\boldsymbol{u}_L, \boldsymbol{v})_\tau - b(\boldsymbol{v}, p_L)_\tau + b(\boldsymbol{u}_L, q)_\tau \\
+\nu \langle \left[\frac{\partial \boldsymbol{u_L}}{\partial \boldsymbol{n}} \right]_A - p_L \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\tau} + \frac{1}{3600 \sigma_\tau \nu} (\boldsymbol{f} - \nabla p_L, \nabla q)_\tau
\end{aligned}
$$

In the sequel of the paper we will need some notion of convergence of the finite element solutions $(\boldsymbol{u_L}, p_L)$ and $(\boldsymbol{u_Q}, p_Q)$ to the weak solution $(\boldsymbol{u}, p)$ of (2.1). In particular, for $\tau \in \mathcal{T}$, we make the following *saturation assumption*

$$
\begin{aligned}
|||(\boldsymbol{u}, p) - (\boldsymbol{u_Q}, p_Q)|||^2 + \left\| h_e^{1/2} \left(\nu \left[\left| \frac{\partial (\boldsymbol{u} - \boldsymbol{u_Q})}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|(p - p_Q)\boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E^2 \\
\leq \beta^2 |||(\boldsymbol{u}, p) - (\boldsymbol{u_L}, p_L)|||^2
\end{aligned} \tag{3.7}
$$

where $\beta = o(1)$, which implies in particular

$$
(1 - \beta)|||(\boldsymbol{e}, \epsilon)||| \leq |||(\boldsymbol{e_Q}, \epsilon_Q)||| \leq (1 + \beta)|||(\boldsymbol{e}, \epsilon)||| \tag{3.8}
$$

This is not a very strong condition since $(\boldsymbol{u_Q}, p_Q)$ is an approximation to $(\boldsymbol{u}, p)$ in a higher degree polynomial space than $(\boldsymbol{u_L}, p_L)$. It supposes however that the solution is more than $\mathcal{H}^1$-regular.

We now present a few basic estimates, which will be used in the next part of this section. Their proofs make use of the following inequalities, valid for $\tau \in \mathcal{T}$ and $e \in E_\tau$:

$$\|\boldsymbol{v}\|_e \leq c_{11} h_\tau^{-1/2} \|\boldsymbol{v}\|_\tau \qquad \boldsymbol{v} \in \mathcal{Q}(\tau) \times \mathcal{Q}(\tau) \tag{3.9}$$

$$\left\| \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right\|_e \leq c_{12} h_\tau^{-1/2} \|\nabla \boldsymbol{v}\|_\tau \qquad \boldsymbol{v} \in \mathcal{Q}(\tau) \times \mathcal{Q}(\tau) \tag{3.10}$$

These inequalities are not satisfied for general functions $\boldsymbol{v}$; they remain valid however for polynomial functions, the constants c_{11} and c_{12} depending then on the degree of the polynomial.

Lemma 3.1.

$$\nu \|\boldsymbol{v}\|_\tau^2 \leq c_{13} h_\tau^2 a(\boldsymbol{v}, \boldsymbol{v})_\tau \qquad \boldsymbol{v} \in \mathcal{K}^2(\tau)$$

Proof. It is a simple reformulation of inequality (2.28) where $\sigma_\tau = \mathcal{O}(h_\tau^{-2})$. □

Lemma 3.2.

$$\sqrt{\nu} \left\| h_e^{-1/2} \boldsymbol{v} \right\|_{\partial\tau} \leq c_{14}\, a(\boldsymbol{v}, \boldsymbol{v})_\tau^{1/2} \qquad \boldsymbol{v} \in \mathcal{K}^2(\tau)$$

Proof. for $e \in \partial\tau$, using inequality (3.9) and Lemma 3.1, we have

$$\nu \|\boldsymbol{v}\|_e^2 \leq c_{11}^2 h_\tau^{-1} \nu \|\boldsymbol{v}\|_\tau^2 \leq c_{13} c_{11}^2 h_\tau\, a(\boldsymbol{v}, \boldsymbol{v})_\tau$$

so that

$$\nu \left\| h_e^{-1/2} \boldsymbol{v} \right\|_{\partial\tau}^2 \leq 3 C_1^{-1} c_{11}^2 c_{13} a(\boldsymbol{v}, \boldsymbol{v})_\tau$$

□

Lemma 3.3. *Assume (3.7) holds. Then*

$$\left\| h_e^{1/2} \left(\nu \left[\left| \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|\epsilon \boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E \leq c_{15} |||(\boldsymbol{e}, \epsilon)|||$$

Proof. The proof is similar to the proof of Lemma 3.1 in [5] with only a few differences due to the pressure term. For $e \in E$ such that $e = \tau_{in} \cap \tau_{out}$

$$\left\| h_e^{1/2} \left(\nu \left[\left| \frac{\partial (\boldsymbol{u_Q} - \boldsymbol{u_L})}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|(p_Q - p_L)\boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_e^2$$

$$= \left\| h_e^{1/2} \left(\nu \left[\left| \frac{\partial \boldsymbol{e_Q}}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|\epsilon_Q \boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_e^2$$

$$
\begin{aligned}
&\leq h_e \left\{ \nu \left\| \frac{\partial \boldsymbol{e_Q}}{\partial \boldsymbol{n}}|_{\tau_{in}} \right\|_e^2 + \frac{\|(\epsilon_Q \boldsymbol{n})|_{\tau_{in}}\|_e^2}{\nu} + \nu \left\| \frac{\partial \boldsymbol{e_Q}}{\partial \boldsymbol{n}}|_{\tau_{out}} \right\|_e^2 \right\} \\
&\leq h_e \left\{ c_{12}^2 \nu (h_{\tau_{in}}^{-1} \|\nabla \boldsymbol{e_Q}\|_{\tau_{in}}^2 + h_{\tau_{out}}^{-1} \|\nabla \boldsymbol{e_Q}\|_{\tau_{out}}^2) + \frac{c_{11}^2}{\nu} h_{\tau_{in}}^{-1} \|\epsilon_Q\|_{\tau_{in}}^2 \right\} \\
&\leq C_2 (c_{12}^2 + c_{11}^2)(|||(\boldsymbol{e_Q}, \epsilon_Q)|||_{\tau_{in}}^2 + |||(\boldsymbol{e_Q}, \epsilon_Q)|||_{\tau_{out}}^2)
\end{aligned}
$$

Therefore, summing over all edges, we get

$$
\begin{aligned}
\left\| h_e^{1/2} \left(\nu \left[\left| \frac{\partial (\boldsymbol{u_Q} - \boldsymbol{u_L})}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|(p_Q - p_L)\boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E^2 &\leq 3C_2 (c_{12}^2 + c_{11}^2) |||(\boldsymbol{e}_Q, \epsilon_Q)|||^2 \\
&\leq 3C_2 (c_{12}^2 + c_{11}^2)(1+\beta)^2 |||(\boldsymbol{e}, \epsilon)|||^2 \qquad (3.11)
\end{aligned}
$$

Finally, combining (3.7), (3.11) and using the triangular inequality we get

$$
\begin{aligned}
&\left\| h_e^{1/2} \left(\nu \left[\left| \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|\epsilon \boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E \\
&\leq \left\| h_e^{1/2} \left(2\nu \left[\left| \frac{\partial (\boldsymbol{u} - \boldsymbol{u}_Q)}{\partial \boldsymbol{n}} \right| \right]_A^2 + 2\nu \left[\left| \frac{\partial \boldsymbol{e}_Q}{\partial \boldsymbol{n}} \right| \right]_A^2 + 2\frac{|(p - p_Q)\boldsymbol{n}|^2}{\nu} + 2\frac{|\epsilon_Q \boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E \\
&\leq \left\| h_e^{1/2} \sqrt{2} \left(\nu \left[\left| \frac{\partial (\boldsymbol{u} - \boldsymbol{u}_Q)}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|(p - p_Q)\boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E \\
&\qquad + \left\| h_e^{1/2} \sqrt{2} \left(\nu \left[\left| \frac{\partial \boldsymbol{e}_Q}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|\epsilon_Q \boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E \\
&\leq \beta \sqrt{2} |||(\boldsymbol{e}, \epsilon)||| + (1+\beta) \sqrt{6C_2 (c_{12}^2 + c_{11}^2)} |||(\boldsymbol{e}, \epsilon)|||
\end{aligned}
$$

□

LEMMA 3.4. *The "continuity" inequality*

$$
|B((\boldsymbol{e}, \epsilon), (\boldsymbol{v}, q))| \leq c_{16} \, |||(\boldsymbol{e}, \epsilon)||| \, |||(\boldsymbol{v}, q)|||
$$

holds for all $(\boldsymbol{v}, q) \in \mathcal{K} \times \mathcal{K} \times \mathcal{K}$.

Proof. We bound successively each term in the definition of $B(\cdot, \cdot)$. The first three terms are easily bounded, namely:

$$
\begin{aligned}
|a(\boldsymbol{e}, \boldsymbol{v})| &\leq \nu \, \|\nabla \boldsymbol{e}\| \, \|\nabla \boldsymbol{v}\| \\
|b(\boldsymbol{v}, \epsilon)| &\leq \|\epsilon\| \, \|\nabla \cdot \boldsymbol{v}\| \leq \sqrt{2} \|\epsilon\| \, \|\nabla \boldsymbol{v}\| \\
\text{and} \quad |b(\boldsymbol{e}, q)| &\leq \|\nabla \cdot \boldsymbol{e}\| \, \|q\| \leq \sqrt{2} \|\nabla \boldsymbol{e}\| \, \|q\|
\end{aligned}
$$

while the two remaining terms need to be integrated by parts; for $\tau \in \mathcal{T}$ we have:

$$
(\nabla \epsilon, \nabla q)_\tau = -(\epsilon, \Delta q)_\tau + \langle \epsilon \boldsymbol{n}, \nabla q \rangle_{\partial \tau}
$$

On one hand we get

$$\left|\sum_{\tau\in\mathcal{T}} \frac{(\epsilon, \Delta q)_\tau}{3600\sigma_\tau\nu}\right| \leq \frac{\sqrt{15}}{\nu}\sum_{\tau\in\mathcal{T}} \|\epsilon\|_\tau \|q\|_\tau \leq \frac{\sqrt{15}}{\nu}\|\epsilon\|\ \|q\|$$

and on the other hand the boundary term gives

$$\begin{aligned}\left|\sum_{\tau\in\mathcal{T}} \frac{\langle \epsilon\boldsymbol{n}, \nabla q\rangle_{\partial\tau}}{3600\sigma_\tau\nu}\right| &= \left|-\frac{1}{3600\nu}\sum_{e\in E}\langle \epsilon, \left[\frac{1}{\sigma_\tau}\frac{\partial q}{\partial \boldsymbol{n}}\right]_J\rangle_e\right| \\ &\leq \frac{1}{3600\nu}\sum_{e\in E}\|\epsilon\|_e \left\|\left[\frac{1}{\sigma_\tau}\frac{\partial q}{\partial \boldsymbol{n}}\right]_J\right\|_e\end{aligned}$$

Now, for $e = \tau_{in}\cap\tau_{out}\in E$:

$$\begin{aligned}\left\|\left[\frac{1}{\sigma_\tau}\frac{\partial q}{\partial \boldsymbol{n}}\right]_J\right\|_e^2 &\leq 2c_{12}^2(h_{\tau_{in}}^{-1}\sigma_{\tau_{in}}^{-2}\|\nabla q\|_{\tau_{in}}^2 + h_{\tau_{out}}^{-1}\sigma_{\tau_{out}}^{-2}\|\nabla q\|_{\tau_{out}}^2) \\ &\leq 18\cdot 10^4\ c_{12}^2 C_4(h_{\tau_{in}}\|q\|_{\tau_{in}}^2 + h_{\tau_{out}}\|q\|_{\tau_{out}}^2) \\ &\leq 18\cdot 10^4\ c_{12}^2 C_4 C_1^{-1} h_e(\|q\|_{\tau_{in}}^2 + \|q\|_{\tau_{out}}^2)\end{aligned}$$

Note that the same inequality holds with the jump of $\frac{1}{\sigma_\tau}\frac{\partial q}{\partial \boldsymbol{n}}$ replaced by its average value.

Consequently

$$\begin{aligned}\left|\sum_{\tau\in\mathcal{T}} \frac{\langle \epsilon\boldsymbol{n}, \nabla q\rangle_{\partial\tau}}{3600\sigma_\tau\nu}\right| &\leq \frac{c_{12}\sqrt{C_4C_1^{-1}}}{6\nu\sqrt{2}}\left(\sum_{e\in E} h_e\|\epsilon\|_e^2\right)^{1/2}\left(3\sum_{\tau\in\mathcal{T}}\|q\|_\tau^2\right)^{1/2} \\ &\leq c_{17}\left\|h_e^{1/2}\frac{|\epsilon|}{\sqrt{\nu}}\right\|_E \frac{\|q\|}{\sqrt{\nu}}\end{aligned}$$

where $c_{17} = \frac{c_{12}\sqrt{C_4C_1^{-1}}}{2\sqrt{6}}$.

Likewise, an integration by parts on the last term yields

$$(\Delta\boldsymbol{e}, \nabla q)_\tau = -(\nabla\boldsymbol{e}, \nabla(\nabla q))_\tau + \langle\frac{\partial\boldsymbol{e}}{\partial\boldsymbol{n}}, \nabla q\rangle_{\partial\tau}$$

and by Lemma 2.1:

$$\left|\sum_{\tau\in\mathcal{T}} \frac{(\nabla\boldsymbol{e}, \nabla(\nabla q))_\tau}{3600\sigma_\tau}\right| \leq \frac{3\sqrt{3}}{\sqrt{5}}\|\nabla\boldsymbol{e}\|\ \|q\|$$

As for the boundary term involving the velocity, a calculation comparable to the one performed above for the pressure implies

$$\begin{aligned}\left|\sum_{\tau\in\mathcal{T}} \frac{\langle\frac{\partial\boldsymbol{e}}{\partial\boldsymbol{n}}, \nabla q\rangle_{\partial\tau}}{3600\sigma_\tau}\right| = \Bigg|-\frac{1}{3600}\sum_{e\in E}\Bigg(&\langle\left[\frac{\partial\boldsymbol{e}}{\partial\boldsymbol{n}}\right]_A, \left[\frac{1}{\sigma_\tau}\nabla q\right]_J\rangle \\ &+\langle\left[\frac{\partial\boldsymbol{e}}{\partial\boldsymbol{n}}\right]_J, \left[\frac{1}{\sigma_\tau}\nabla q\right]_A\rangle\Bigg)\Bigg|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{c_{17}}{\sqrt{3}} \sum_{e\in E} h_e^{1/2} \left(\left\| \left[\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right]_A \right\|_e + \left\| \left[\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right]_J \right\|_e \right) (\|q\|^2_{\tau_{in}} + \|q\|^2_{\tau_{out}})^{1/2} \\
&\leq c_{17} \sum_{e\in E} 3h_e^{1/2} \left\| \left[\left| \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right| \right]_A \right\|_e (\|q\|^2_{\tau_{in}} + \|q\|^2_{\tau_{out}})^{1/2} \\
&\leq 3c_{17} \left\| \sqrt{\nu} h_e^{1/2} \left[\left| \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right| \right]_A \right\|_E \frac{\|q\|}{\sqrt{\nu}}
\end{aligned}
$$

Adding all the terms together finally yields:

$$
\begin{aligned}
&|B((\boldsymbol{e},\epsilon))| \leq \nu \|\nabla \boldsymbol{e}\| \, \|\nabla \boldsymbol{v}\| + \sqrt{2} \, \|\epsilon\| \, \|\nabla \boldsymbol{v}\| \\
&+\sqrt{2} \, \|\nabla \boldsymbol{e}\| \, \|q\| + \frac{\sqrt{15}}{\nu} \|\epsilon\| \, \|q\| + \frac{3\sqrt{3}}{\sqrt{5}} \|\nabla \boldsymbol{e}\| \, \|q\| \\
&+c_{17} \left\| h_e^{1/2} \frac{|\epsilon|}{\sqrt{\nu}} \right\|_E \frac{\|q\|}{\sqrt{\nu}} + 3c_{17} \left\| \sqrt{\nu} h_e^{1/2} \left[\left| \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right| \right]_A \right\|_E \frac{\|q\|}{\sqrt{\nu}} \\
&\leq \left(\nu \|\nabla \boldsymbol{v}\|^2 + \frac{\|q\|^2}{\nu} \right)^{1/2} \left(\frac{42}{5} \nu \|\nabla \boldsymbol{e}\|^2 + 17 \frac{\|\epsilon\|^2}{\nu} \right. \\
&\left. +9c_{17}^2 \left(\left\| h_e^{1/2} \frac{|\epsilon|}{\sqrt{\nu}} \right\|_E^2 + \left\| \sqrt{\nu} h_e^{1/2} \left[\left| \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right| \right]_A \right\|_E^2 \right) \right)^{1/2} \\
&\leq |||(\boldsymbol{v},q)||| \, (17|||(\boldsymbol{e},\epsilon)|||^2 + 9c_{17}^2 c_{15} |||(\boldsymbol{e},\epsilon)|||^2)^{1/2} \\
&\leq \sqrt{17 + 9c_{17}^2 c_{15}} \, |||(\boldsymbol{e},\epsilon)||| \, |||(\boldsymbol{v},q)|||
\end{aligned}
$$

□

LEMMA 3.5. *Let* $(\boldsymbol{v},q) \in \mathcal{K}^3$ *and* $(\boldsymbol{\chi},\rho) \in \mathcal{L}^3$. *Then there exists a positive constant* $\gamma < 1$ *such that*

$$
|||(\boldsymbol{v},q)||| \leq \frac{1}{\sqrt{1-\gamma^2}} |||(\boldsymbol{v}+\boldsymbol{\chi}, q+\rho)|||
$$

Proof. Let $\tau \in \mathcal{T}$ with angles $\theta_i, i = 1,3$. Define $d_\tau = cos^2\theta_{1,\tau} + cos^2\theta_{2,\tau} + cos^2\theta_{3,\tau}$. From [11] we know that

$$
(\nabla \boldsymbol{v}, \nabla \chi)_\tau \leq \gamma_\tau \|\nabla \boldsymbol{v}\|_\tau \|\nabla \boldsymbol{\chi}\|_\tau
$$

and

$$
(q,\rho)_\tau \leq \frac{\sqrt{15}}{4} \|q\|_\tau \|\rho\|_\tau
$$

with $\gamma_\tau^2 = \frac{1}{2} + \frac{1}{3}\sqrt{d_\tau - \frac{3}{4}}$. If the triangulation satisfies a minimal angle condition, then $d_\tau < 3$ and thus $\gamma \equiv \max(\max_{\tau\in\mathcal{T}} \gamma_\tau, \frac{\sqrt{15}}{4}) < 1$. (For example, when the minimal angle is 5 degrees we have $\gamma \simeq 0.99873$). Next we get

$$
|((\boldsymbol{v},q),(\boldsymbol{\chi},\rho))| \equiv \left| \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\chi}) + \frac{(q,\rho)}{\nu} \right| \leq \gamma |||(\boldsymbol{v},q)||| \, |||(\boldsymbol{\chi},\rho)|||
$$

and finally

$$
\begin{aligned}
|||(\boldsymbol{v}+\boldsymbol{\chi}, q+\rho)|||^2 &= |||(\boldsymbol{v}, q)|||^2 + |||(\boldsymbol{\chi}, \rho)|||^2 + 2((\boldsymbol{v}, q), (\boldsymbol{\chi}, \rho)) \\
&\geq |||(\boldsymbol{v}, q)|||^2 + |||(\boldsymbol{\chi}, \rho)|||^2 - 2\gamma |||(\boldsymbol{v}, q)|||\ |||(\boldsymbol{\chi}, \rho)||| \\
&\geq (1-\gamma^2)|||(\boldsymbol{v}, q)|||^2
\end{aligned}
$$

□ The following theorem proves that the estimate $(\check{\boldsymbol{e}}, \check{\epsilon})$ is a global upper and lower bound of the error $(\boldsymbol{e}, \epsilon)$ in the $|||\cdot|||$ norm.

THEOREM 3.6. *there exist two positive constants c_{18} and c_{19} such that*

$$
c_{18}|||(\boldsymbol{e}, \epsilon)||| \leq |||(\check{\boldsymbol{e}}, \check{\epsilon})||| \leq c_{19}|||(\boldsymbol{e}, \epsilon)||| \tag{3.12}
$$

Proof. Let $(\boldsymbol{e}_Q^L, \epsilon_Q^L)$ and $(\boldsymbol{e}_Q^Q, \epsilon_Q^Q)$ be respectively the linear and (continuous) quadratic parts of $(\boldsymbol{e}_Q, \epsilon_Q)$, so that

$$
(\boldsymbol{e}_Q, \epsilon_Q) = (\boldsymbol{e}_Q^L, \epsilon_Q^L) + (\boldsymbol{e}_Q^Q, \epsilon_Q^Q)
$$

Recalling (3.5), (3.6) and using (3.3), we have

$$
\begin{aligned}
B((\check{\boldsymbol{e}}, \check{\epsilon}), (\boldsymbol{e}_Q^Q, \epsilon_Q^Q)) &= B((\boldsymbol{e}_{\boldsymbol{Q}}, \epsilon_Q), (\boldsymbol{e}_Q^Q, \epsilon_Q^Q)) \\
&= B((\boldsymbol{e}_{\boldsymbol{Q}}, \epsilon_Q), (\boldsymbol{e}_Q, \epsilon_Q))
\end{aligned}
$$

Now we have, from Theorem 2.2 and Lemmas 2.4 and 3.5:

$$
\begin{aligned}
|||(\boldsymbol{e}_Q, \epsilon_Q)|||^2 &\leq \beta_4^2\ \|(\boldsymbol{e}_Q, \epsilon_Q)\|_*^2 \\
&\leq \beta_4^2 \omega^{-1}\ B((\boldsymbol{e}_Q, \epsilon_Q), (\boldsymbol{e}_Q, \epsilon_Q)) \\
&= \beta_4^2 \omega^{-1}\ B((\check{\boldsymbol{e}}, \check{\epsilon}), (\boldsymbol{e}_Q^Q, \epsilon_Q^Q)) \\
&\leq 27\beta_4^2 \omega^{-1}\ |||(\check{\boldsymbol{e}}, \check{\epsilon})|||\ |||(\boldsymbol{e}_Q^Q, \epsilon_Q^Q)||| \\
&\leq \frac{27\beta_4^2\omega^{-1}}{\sqrt{1-\gamma^2}} |||(\check{\boldsymbol{e}}, \check{\epsilon})||| \cdot |||(\boldsymbol{e}_Q, \epsilon_Q)|||
\end{aligned}
$$

and thus

$$
(1-\beta)\ |||(\boldsymbol{e}, \epsilon)||| \leq |||(\boldsymbol{e}_Q, \epsilon_Q)||| \leq \frac{27\beta_4^2\omega^{-1}}{\sqrt{1-\gamma^2}} |||(\check{\boldsymbol{e}}, \check{\epsilon})|||
$$

Conversely we have

$$
\begin{aligned}
|||(\check{\boldsymbol{e}}, \check{\epsilon})|||^2 &\leq \beta_2^2\ \|(\check{\boldsymbol{e}}, \check{\epsilon})\|_*^2\ \leq\ \beta_2^2\omega^{-1}\ B((\check{\boldsymbol{e}}, \check{\epsilon}), (\check{\boldsymbol{e}}, \check{\epsilon})) \\
&= \beta_2^2\omega^{-1}\ (B((\boldsymbol{e}, \epsilon), (\check{\boldsymbol{e}}, \check{\epsilon})) - J((\boldsymbol{e}, \epsilon), (\check{\boldsymbol{e}}, \check{\epsilon})))
\end{aligned}
$$

The jump term on the right-hand side is bounded using Lemmas 3.2 and 3.3:

$$
\begin{aligned}
|J((\boldsymbol{e}, \epsilon), (\check{\boldsymbol{e}}, \check{\epsilon}))| &= \left| \langle \nu \left[\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right]_A - \epsilon\ \boldsymbol{n}, [\check{\boldsymbol{e}}]_J \rangle_E \right| \\
&\leq \left\| h_e^{1/2} \left(\nu \left[\left| \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{n}} \right| \right]_A^2 + \frac{|\epsilon\ \boldsymbol{n}|^2}{\nu} \right)^{1/2} \right\|_E \|h_e^{-1/2}\sqrt{\nu}[\check{\boldsymbol{e}}]_J\|_E \\
&\leq c_{15}\ |||(\boldsymbol{e}, \epsilon)|||\ \sqrt{\nu} \left(\sum_{e \in E} \|h_e^{-1/2}[\check{\boldsymbol{e}}]_J\|_e^2 \right)^{1/2} \\
&\leq \sqrt{2} c_{15} c_{14}\ |||(\boldsymbol{e}, \epsilon)|||\ |||(\check{\boldsymbol{e}}, \check{\epsilon})|||
\end{aligned}
$$

Finally, by Lemma 3.4, we have

$$|||(\check{\boldsymbol{e}},\check{\epsilon})|||^2 \leq \beta_2^2\omega^{-1}(27|||(\boldsymbol{e}_Q,\epsilon_Q)||| \; |||(\check{\boldsymbol{e}},\check{\epsilon})||| + c_{16}|||(\boldsymbol{e},\epsilon)||| \; |||(\check{\boldsymbol{e}},\check{\epsilon})|||)$$

and hence

$$|||(\check{\boldsymbol{e}},\check{\epsilon})||| \leq \beta_2^2\omega^{-1}(27(1+\beta)+c_{16})|||(\boldsymbol{e},\epsilon)|||$$

□

Let $\boldsymbol{e}^{'} = \boldsymbol{u} - \boldsymbol{u}_{h,l}$ and $\epsilon^{'} = p - p_h$. We now define the *local* error estimate $(\tilde{\boldsymbol{e}},\tilde{\epsilon}) \in \mathcal{K}^3(\tau)$ by

$$B((\tilde{\boldsymbol{e}},\tilde{\epsilon}),(\boldsymbol{v},q)) = L^{'}((\boldsymbol{v},q)) \qquad (\boldsymbol{v},q) \in \mathcal{K}^3(\tau) \tag{3.13}$$

with

$$\begin{aligned} L^{'}((\boldsymbol{v},q)) = (\boldsymbol{f},\boldsymbol{v})_\tau - a(\boldsymbol{u}_{h,l},\boldsymbol{v})_\tau + (p_h,\nabla.\boldsymbol{v})_\tau - (\nabla.\boldsymbol{u}_{h,l},q)_\tau \\ +\nu\langle\left[\frac{\partial \boldsymbol{u_{h,l}}}{\partial \boldsymbol{n}}\right]_A - p_h\boldsymbol{n},\boldsymbol{v}\rangle_{\partial\tau} + \frac{1}{3600\sigma_\tau\nu}(\boldsymbol{f}-\nabla p_h,\nabla q)_\tau \end{aligned}$$

Lemma 3.7. *For all* $(\boldsymbol{v},q) \in \mathcal{Q}^3$ *we have*

$$\sum_{\tau\in\mathcal{T}}\left|L((\boldsymbol{v},q))_\tau - L^{'}((\boldsymbol{v},q))_\tau\right| \leq c_{20}|||(\boldsymbol{u}_L,p_L)-(\boldsymbol{u}_{h,l},p_h)||| \; |||(\boldsymbol{v},q)||| \tag{3.14}$$

Proof. We write the difference between the right-hand sides of (3.6) and (3.13) in triangle $\tau \in \mathcal{T}$ as

$$\begin{aligned} L((\boldsymbol{v},q))_\tau - L^{'}((\boldsymbol{v},q))_\tau = -a(\boldsymbol{u}_L - \boldsymbol{u_{h,l}},\boldsymbol{v})_\tau + (p_L - p_h,\nabla.\boldsymbol{v})_\tau \\ -(\nabla.(\boldsymbol{u}_L - \boldsymbol{u_{h,l}}),q)_\tau + \langle\nu\left[\frac{\partial(\boldsymbol{u}_L - \boldsymbol{u_{h,l}})}{\partial\boldsymbol{n}}\right]_A - (p_L-p_h)\boldsymbol{n},\boldsymbol{v}\rangle_{\partial\tau} \\ -\frac{1}{3600\sigma_\tau\nu}(\nabla(p_L-p_h),\nabla q)_\tau \end{aligned}$$

Each of these terms is now bounded in terms of the energy norm of the difference $(\boldsymbol{u}_L - \boldsymbol{u}_{h,l}, p_L - p_h)$ between the solution from the Petrov-Galerkin and mini-element formulations, as in the proof of Lemma 3.4: First

$$|a(\boldsymbol{u}_L - \boldsymbol{u_{h,l}},\boldsymbol{v})_\tau| \leq \sqrt{\nu}\|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau a(\boldsymbol{v},\boldsymbol{v})_\tau^{1/2}$$

$$\begin{aligned} |(p_L - p_h,\nabla.\boldsymbol{v})_\tau| &\leq \|p_L - p_h\|_\tau\|\nabla.\boldsymbol{v}\|_\tau \\ &\leq \sqrt{2}\|p_L - p_h\|_\tau\|\nabla\boldsymbol{v}\|_\tau \end{aligned}$$

and similarly

$$|(\nabla.(\boldsymbol{u}_L - \boldsymbol{u_{h,l}}),q)_\tau| \leq \sqrt{2}\|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau\|q\|_\tau$$

The bound on the boundary term is then derived using Lemma 3.2:

$$\left|\langle\nu\left[\frac{\partial(\boldsymbol{u}_L - \boldsymbol{u_{h,l}})}{\partial\boldsymbol{n}}\right]_A,\boldsymbol{v}\rangle_{\partial\tau}\right| \leq \nu\left\|h_e^{1/2}\left[\frac{\partial(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})}{\partial\boldsymbol{n}}\right]_A\right\|_{\partial\tau}\left\|h_e^{-1/2}\boldsymbol{v}\right\|_{\partial\tau}$$

for $e \in \partial\tau$ we have

$$\left\| h_e^{1/2} \left[\frac{\partial(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})}{\partial \boldsymbol{n}} \right]_A \right\|_e^2 \leq \frac{h_e}{2} \left\{ \left\| \frac{\partial(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})}{\partial \boldsymbol{n}}|_\tau \right\|_e^2 + \left\| \frac{\partial(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})}{\partial \boldsymbol{n}}|_{\tau_{out}} \right\|_e^2 \right\}$$
$$\leq \frac{h_e c_{12}^2}{2} \left\{ h_\tau^{-1} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau^2 + h_{\tau_{out}}^{-1} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_{\tau_{out}}^2 \right\}$$
$$\leq \frac{C_2 c_{12}^2}{2} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_{\tau \cup \tau_{out}}^2$$

Thus summing over edges in τ

$$\left\| h_e^{1/2} \left[\frac{\partial(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})}{\partial \boldsymbol{n}} \right]_A \right\|_{\partial\tau}^2 \leq C_2 c_{12}^2 \left(\frac{3}{2} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau^2 + \frac{1}{2} \sum_{\tau'} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_{\tau'}^2 \right)$$

where τ' is a triangle neighbor of τ (total of 3 neighbors in the 2D case). Similarly

$$|\langle (p_L - p_h)\boldsymbol{n}, \boldsymbol{v} \rangle_e| \leq \|h_e^{1/2}(p_L - p_h)\|_e \, \|h_e^{-1/2} \boldsymbol{v}\|_e$$
$$\leq C_2^{1/2} c_{11} \frac{\|p_L - p_h\|_\tau}{\sqrt{\nu}} c_{14} \, a(\boldsymbol{v}, \boldsymbol{v})_\tau^{1/2}$$

Finally, summing over all triangles, we get

$$\sum_{\tau \in \mathcal{T}} \left| L((\boldsymbol{v}, q))_\tau - L'((\boldsymbol{v}, q))_\tau \right| \leq \sqrt{\nu} \sum_{\tau \in \mathcal{T}} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau a(\boldsymbol{v}, \boldsymbol{v})_\tau^{1/2}$$
$$+ \sum_{\tau \in \mathcal{T}} \sqrt{2} \|p_L - p_h\|_\tau \|\nabla \boldsymbol{v}\|_\tau + \sum_{\tau \in \mathcal{T}} \sqrt{2} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau \|q\|_\tau + \frac{25}{\nu} \sum_{\tau \in \mathcal{T}} \|p_L - p_h\|_\tau \, \|q\|_\tau$$
$$+ \sum_{\tau \in \mathcal{T}} C_2^{1/2} c_{12} \left(\frac{3}{2} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau^2 + \frac{1}{2} \sum_{\tau'} \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_{\tau'}^2 \right)^{1/2} c_{14} \sqrt{\nu} \, a(\boldsymbol{v}, \boldsymbol{v})_\tau^{1/2}$$
$$+ \sum_{\tau \in \mathcal{T}} 3 C_2^{1/2} c_{14} c_{11} \|p_L - p_h\|_\tau \, a(\boldsymbol{v}, \boldsymbol{v})_\tau^{1/2}$$
$$\leq \left(\sum_{\tau \in \mathcal{T}} (a(\boldsymbol{v}, \boldsymbol{v})_\tau + \nu^{-1} \|q\|_\tau^2) \right)^{1/2} \cdot$$
$$\left((3 + 2C_2 c_{14}^2 c_{12}^2) \sum_{\tau \in \mathcal{T}} \nu \|\nabla(\boldsymbol{u}_L - \boldsymbol{u}_{h,l})\|_\tau^2 + (627 + 9C_2 c_{14}^2 c_{11}^2) \sum_{\tau \in \mathcal{T}} \frac{\|p_L - p_h\|_\tau^2}{\nu} \right)^{1/2}$$
$$\leq c_{20} |||(\boldsymbol{v}, q)||| \; |||(\boldsymbol{u}_L, p_L) - (\boldsymbol{u}_{h,l}, p_h)|||$$

□

This estimate of the difference between the two right-hand sides of the systems defining the two error estimates $(\tilde{\boldsymbol{e}}, \tilde{\epsilon})$ and $(\check{\boldsymbol{e}}, \check{\epsilon})$ allows us to get a bound for the difference between the estimators themselves, since both systems have the same left-hand side which is positive definite on the space $\mathcal{K}^3$. Indeed, for $(\boldsymbol{v}, q) = (\tilde{\boldsymbol{e}}, \tilde{\epsilon}) - (\check{\boldsymbol{e}}, \check{\epsilon}) \in \mathcal{Q}^3$ in the previous lemma, we have:

$$|||(\tilde{\boldsymbol{e}}, \tilde{\epsilon}) - (\check{\boldsymbol{e}}, \check{\epsilon})|||^2 \leq \beta_2^2 \|(\tilde{\boldsymbol{e}}, \tilde{\epsilon}) - (\check{\boldsymbol{e}}, \check{\epsilon})\|_*^2$$

$$
\begin{aligned}
&\leq \beta_2^2\omega^{-1}\sum_{\tau\in\mathcal{T}} B((\tilde{\boldsymbol{e}},\tilde{\epsilon})-(\check{\boldsymbol{e}},\check{\epsilon}),(\tilde{\boldsymbol{e}},\tilde{\epsilon})-(\check{\boldsymbol{e}},\check{\epsilon}))\\
&= \beta_2^2\omega^{-1}\sum_{\tau\in\mathcal{T}} \Big(L((\tilde{\boldsymbol{e}},\tilde{\epsilon})-(\check{\boldsymbol{e}},\check{\epsilon}))_\tau - L^{'}((\tilde{\boldsymbol{e}},\tilde{\epsilon})-(\check{\boldsymbol{e}},\check{\epsilon}))_\tau\Big)\\
&\leq \beta_2^2\omega^{-1}c_{20}|||(\tilde{\boldsymbol{e}},\tilde{\epsilon})-(\check{\boldsymbol{e}},\check{\epsilon})|||\ |||(\boldsymbol{u}_L,p_L)-(\boldsymbol{u}_{h,l},p_h)|||
\end{aligned}
$$

Noting that $(\boldsymbol{u}_L,p_L)-(\boldsymbol{u}_{h,l},p_h)=(\boldsymbol{e},\epsilon)-(\boldsymbol{e}^{'},\epsilon^{'})$ and recalling Theorem 2.5 we thus have the

THEOREM 3.8.

$$
|||(\tilde{\boldsymbol{e}},\tilde{\epsilon})-(\check{\boldsymbol{e}},\check{\epsilon})||| \leq c_{14}|||(\boldsymbol{e},\epsilon)-(\boldsymbol{e}^{'},\epsilon^{'})||| \leq \frac{c_{14}c_{10}h^2}{\sqrt{\nu}}\|\nabla\boldsymbol{f}\| \tag{3.15}
$$

Note that if neither $(\boldsymbol{u}_L,p_L)$ nor $(\boldsymbol{u}_{h,l},p_h)$ is the exact solution to the original Stokes system, then both terms $(\boldsymbol{e},\epsilon)$ and $(\boldsymbol{e}^{'},\epsilon^{'})$ are of order $\mathcal{O}(h)$ and their difference is of order $\mathcal{O}(h^2)$. However, when for example $(\boldsymbol{u}_L,p_L)$ is the exact solution (which is possible for properly chosen functions $\boldsymbol{f}$ and $\boldsymbol{g}$ and a particular domain Ω), $(\boldsymbol{e},\epsilon)$ is zero while $(\boldsymbol{e}^{'},\epsilon^{'})$ is of order $\mathcal{O}(h^2)$. Therefore there is no constant c_{21} such that

$$
c_{21}^{-1}(h)|||(\boldsymbol{e},\epsilon)||| \leq |||(\boldsymbol{e}^{'},\epsilon^{'})||| \leq c_{21}(h)|||(\boldsymbol{e},\epsilon)|||
$$

Finally the best result we can get reads

THEOREM 3.9. *There exist two positive constants c_{22} and c_{23} such that*

$$
c_{22}|||(\boldsymbol{e}^{'},\epsilon^{'})||| - \mathcal{O}(h^2) \leq |||(\tilde{\boldsymbol{e}},\tilde{\epsilon})||| \leq c_{23}|||(\boldsymbol{e}^{'},\epsilon^{'})||| + \mathcal{O}(h^2) \tag{3.16}
$$

Proof. It is a direct consequence of Theorems 3.6 and 3.8. □

In the last step of our analysis we simplify the system (3.13). We introduce the final estimate $(\boldsymbol{e}^{''},\epsilon^{''})\in\mathcal{K}^3$ defined in triangle τ by

$$
\left\{
\begin{array}{lcl}
a(\boldsymbol{e}^{''},\boldsymbol{v})_\tau + b(\boldsymbol{v},\epsilon^{''})_\tau & = & (\boldsymbol{f},\boldsymbol{v})_\tau - a(\boldsymbol{u}_{h,l},\boldsymbol{v})_\tau + (p_h,\nabla.\boldsymbol{v})_\tau\\
 & & +\langle\nu\left[\dfrac{\partial\boldsymbol{u}_{h,l}}{\partial\boldsymbol{n}}\right]_A - p_h\boldsymbol{n},\boldsymbol{v}\rangle_{\partial\tau}\\
b(\boldsymbol{e}^{''},q)_\tau - \dfrac{(\nabla\epsilon^{''},\nabla q)_\tau}{3600\sigma_\tau\nu} & = & (\nabla.\boldsymbol{u}_{h,l},q)_\tau - \dfrac{1}{3600\sigma_\tau\nu}(\boldsymbol{f}-\nabla p_h,\nabla q)_\tau
\end{array}
\right. \tag{3.17}
$$

for $(\boldsymbol{v},q)\in\mathcal{K}^3$.

Note that the right-hand side has not changed and that only the Laplace term has been removed from the left-hand side, when compared to the system (3.13).

The corresponding 9×9 matrix M_τ of this system takes the form

$$
M_\tau = \begin{pmatrix} A_\tau & 0 & B_{\tau,x}^t\\ 0 & A_\tau & B_{\tau,y}^t\\ B_{\tau,x} & B_{\tau,y} & -S_\tau \end{pmatrix} \tag{3.18}
$$

where the 3×3 matrices A_τ, $B_{\tau,x}$ and $B_{\tau,y}$ are respectively defined by

$$(A_\tau)_{i,j} = \nu \int_\tau \nabla\psi_{ib}(\tau)\nabla\psi_{ib}(\tau)d\tau \qquad 1 \le i,j \le 3 \tag{3.19}$$

$$(B_{\tau,x})_{i,j} = -\int_\tau \psi_{jb}(\tau)\frac{\partial\psi_{ib}(\tau)}{\partial x}d\tau \quad 1 \le i,j \le 3 \tag{3.20}$$

$$(B_{\tau,y})_{i,j} = -\int_\tau \psi_{jb}(\tau)\frac{\partial\psi_{ib}(\tau)}{\partial y}d\tau \quad 1 \le i,j \le 3 \tag{3.21}$$

($\psi_{1b} = 4\psi_2\psi_3$ and cyclically) and $S_\tau = \dfrac{1}{3600\sigma_\tau\nu^2}A_\tau$.

In the case of a boundary triangle, the equations in the system (3.17) associated with the corresponding edge(s) are replaced by a scaled version of the Dirichlet boundary condition $\boldsymbol{e}'' = \boldsymbol{u} - \boldsymbol{u}^*_{h,l}$, where $\boldsymbol{u}^*_{h,l}$ represents the linearly interpolated value of $\boldsymbol{u}_{h,l}$ at the midpoint of the edge. Thus all elements but the diagonal terms in the corresponding rows of M_τ are zeroed out.

Since $\mathcal{K}(\tau)$ does not contain the constant functions, the matrix A_τ is symmetric positive definite. If τ is an interior triangle, then the Schur complement $C_\tau = C'_\tau + S_\tau$ of M_τ, with $C'_\tau \equiv B_{\tau,x}A_\tau^{-1}B^t_{\tau,x} + B_{\tau,y}A_\tau^{-1}B^t_{\tau,y}$, is therefore well defined, and because S_τ is positive definite, C_τ is also positive definite. The matrix C'_τ can in fact be shown to be independent of the geometry of the triangle τ, even though the matrices A_τ, $B_{\tau,x}$ and $B_{\tau,y}$ are not (see [17]).

In a boundary element, the matrix C'_τ does now depend on the geometry of the element, but is still positive semi-definite, so that C_τ is non-singular.

Hence the matrix M_τ is non singular since

$$\begin{pmatrix} A_\tau & 0 & B^t_{\tau,x} \\ 0 & A_\tau & B^t_{\tau,y} \\ B_{\tau,x} & B_{\tau,y} & -S_\tau \end{pmatrix} = \begin{pmatrix} A_\tau & 0 & 0 \\ 0 & A_\tau & 0 \\ B_{\tau,x} & B_{\tau,y} & -C_\tau \end{pmatrix} \begin{pmatrix} I & 0 & A_\tau^{-1}B^t_{\tau,x} \\ 0 & I & A_\tau^{-1}B^t_{\tau,y} \\ 0 & 0 & I \end{pmatrix} \tag{3.22}$$

so that the system (3.17) has a unique solution (the non-singularity of the Schur complement C_τ (and hence its positive definiteness), together with the non-singularity of A_τ, is equivalent to the stability of the discretization of the error, i.e. the discretization will satisfy a local Babuška-Brezzi type condition).

In our final theorem we compare this last estimate with the discretization error resulting from the mini-element formulation.

THEOREM 3.10. *There exist two positive constants c_{24} and c_{25} such that*

$$c_{24}|||(\boldsymbol{e}'', \epsilon'')||| \le |||(\tilde{\boldsymbol{e}}, \tilde{\epsilon})||| \le c_{25}|||(\boldsymbol{e}'', \epsilon'')|||$$

Proof.

$$\begin{aligned} |||(\boldsymbol{e}'', \epsilon'')|||^2 &\le \beta_2^2 \|(\boldsymbol{e}'', \epsilon'')\|_*^2 \\ &= \beta_2^2 \left(a(\boldsymbol{e}'', \boldsymbol{e}'') + \sum_{\tau\in\mathcal{T}} \frac{1}{3600\sigma_\tau\nu}\|\nabla\epsilon''\|_\tau^2 \right) \\ &= \beta_2^2\, L'((\boldsymbol{e}'', \epsilon'')) \\ &= \beta_2^2\, B((\tilde{\boldsymbol{e}}, \tilde{\epsilon}), (\boldsymbol{e}'', \epsilon'')) \\ &\le 27\beta_2^2\, |||(\tilde{\boldsymbol{e}}, \tilde{\epsilon})|||\, |||(\boldsymbol{e}'', \epsilon'')||| \end{aligned}$$

Conversely,

$$
\begin{aligned}
|||(\tilde{\boldsymbol{e}},\tilde{\epsilon})|||^2 &\leq \beta_2^2 \, \|(\tilde{\boldsymbol{e}},\tilde{\epsilon})\|_*^2 \\
&\leq \beta_2^2 \omega^{-1} B((\tilde{\boldsymbol{e}},\tilde{\epsilon}),(\tilde{\boldsymbol{e}},\tilde{\epsilon})) = \beta_2^2 \omega^{-1} L'((\tilde{\boldsymbol{e}},\tilde{\epsilon})) \\
&\leq \beta_2^2 \omega^{-1} \left(a(\boldsymbol{e}^{''},\tilde{\boldsymbol{e}}) + \sum_{\tau\in\mathcal{T}} \frac{1}{3600\sigma_\tau \nu} (\nabla\epsilon^{''},\nabla\tilde{\epsilon})_\tau + b(\tilde{\boldsymbol{e}},\epsilon^{''}) - b(\boldsymbol{e}^{''},\tilde{\epsilon}) \right) \\
&\leq \beta_2^2 \omega^{-1} \left(\|(\boldsymbol{e}^{''},\epsilon^{''})\|_* \|(\tilde{\boldsymbol{e}},\tilde{\epsilon})\|_* + \sqrt{2}\|\epsilon^{''}\| \, \|\nabla\tilde{\boldsymbol{e}}\| + \sqrt{2}\|\tilde{\epsilon}\| \, \|\nabla\boldsymbol{e}^{''}\| \right) \\
&\leq \beta_2^2 \omega^{-1} (\sqrt{2} + \beta_1^{-2}) |||(\boldsymbol{e}^{''},\epsilon^{''})||| \, |||(\tilde{\boldsymbol{e}},\tilde{\epsilon})|||
\end{aligned}
$$

□

This equivalence between the two estimate is true locally as well, since all inequalities in the proof remain valid in any triangle $\tau \in \mathcal{T}$.

COROLLARY 3.11. *There exist positive constants c_{26}, c_{27}, c_{28} and c_{29} such that*

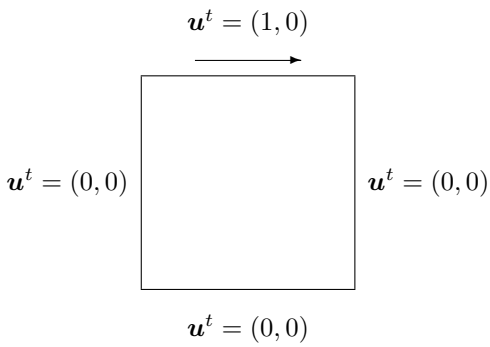
$$
c_{26}|||(\boldsymbol{e}^{'},\epsilon^{'})||| - c_{27}h^2 \leq |||(\boldsymbol{e}^{''},\epsilon^{''})||| \leq c_{28}|||(\boldsymbol{e}^{'},\epsilon^{'})||| + c_{29}h^2
$$

This last result means that the error estimate $(\boldsymbol{e}^{''},\epsilon^{''})$ is a reasonable global estimate of the discretization error resulting from the use of the mini-element. No local inequality is proved. However, as will be seen in section 4, this estimate seems to be also a good local indicator of the size of the error and becomes an efficient tool in the adaptation process of the mesh to the solution of the original problem (2.1).

4. Numerical Results. The system (3.17) is solved in each triangle τ using the decomposition (3.22). Thus we need to solve *eight* 3×3 systems with the matrix A_τ (among which *six* are necessary to compute $A_\tau^{-1} B_{\tau,x}^t$ and $A_\tau^{-1} B_{\tau,x}^t$), and then *another* 3×3 system with C_τ to get the pressure error and finally the velocity error.

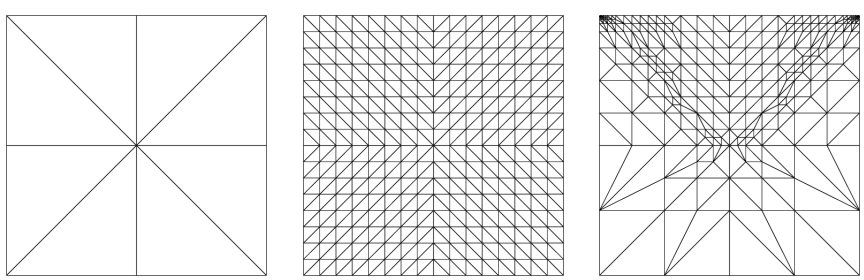
We present three numerical examples which demonstrate the efficiency of our error estimator in controlling the mesh adaptation process and also in estimating the discretization error.

4.1. Driven Cavity. We consider first the standard case in CFD of a driven cavity in a unit square. In this example $\nu = 0.01$. The right hand side $\boldsymbol{f}$ in (2.1) is set to zero and boundary conditions are zero except on the upper side of the square where the velocity is tangential and has value 1:



This domain is first triangulated into $NT = 8$ triangles, to form the level 1 grid (Fig. 4.1**(a)**). After computing the solution of (2.1) on this coarse grid, the mesh is either uniformly (Fig. 4.1**(b)**) or adaptively (Fig. 4.1**(c)**) refined. Each triangle is subdivided into 4 smaller triangles by joining the midpoints of the three sides (regular refinement). "Green" triangles are added to make the mesh an admissible triangulation. Refinement stops when the total number of vertices NV in the mesh reaches a preassigned target value. For further details on the refinement strategy the reader can refer to [4].

We can notice that the adaptive strategy created a lot of triangles (up to 21 levels) in the two upper corners of the cavity, where two singularities arise due to the discontinuities in the boundary conditions.



(a) **(b)** **(c)**

FIG. 4.1. **(a)** *Initial grid with $NV = 9$ vertices and $NT = 8$ triangles;* **(b)** *uniformly refined grid with $NV = 289$ vertices and $NT = 512$ triangles (4 levels) and* **(c)** *adaptively refined grid with $NV = 282$ vertices and $NT = 496$ triangles (10 levels).*

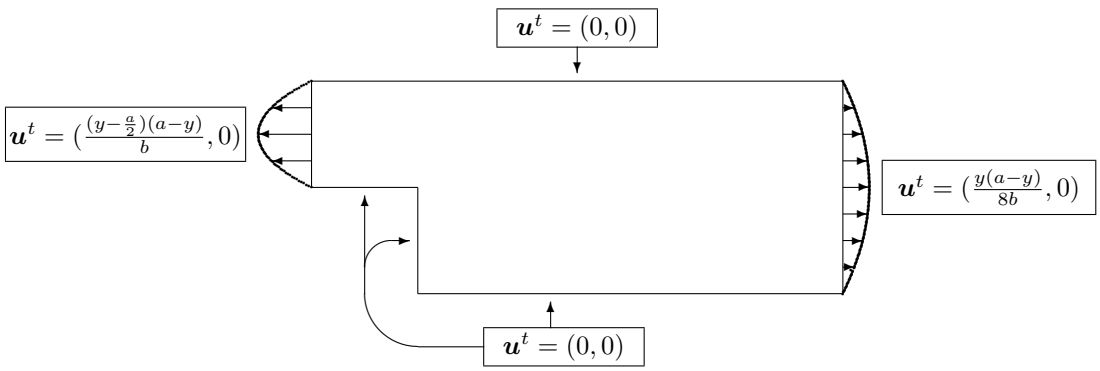
The resulting velocity field is plotted on Figure 4.2. Each arrow represents the velocity at the inner vertices, with a streamline direction and a length proportional to the norm of the velocity. Although the picture seems to be nicer for the uniform refinement case because of the equal spacing of the vertices, the one corresponding to the adaptive strategy gives a better idea of the real field, especially at the two top corners: on the left the fluid is taken away and is forced back into the cavity on the

right side, hence the corresponding negative and positive pressures on Figures 4.3**(a)** and **(b)**.

The capture of the discontinuities is much more effective on the adapted mesh than on the uniform mesh, and the level of the pressure is about 10^4 times the level reached on the uniform grid.

Error estimates are shown on Figures 4.4**(a)** and **(b)**. In order to draw continuous functions, errors at the vertices are computed by averaging the estimates on neighboring triangles.

4.2. Backward Facing Step. Another classical case in CFD is the backward facing step. Here the spikes in the pressure are not due to a discontinuity in the boundary conditions but only to a discontinuity in the geometry of the boundary. Boundary conditions are as follows:



The initial grid (Fig. 4.5**(a)**) is uniformly refined (Fig. 4.5**(b)**, 4 levels of refinement) or adaptively refined (Fig. 4.5**(c)**, 9 levels of refinement) based on the error estimate computed from the solution ($\nu = 1$). In the latter case triangles were added near the re-entrant corner to resolve the discontinuity. Profiles of the velocity field are very similar this time (Fig. 4.6,4.7) but the drop in pressure is better represented with the adaptive strategy (Fig. 4.8). Note that in Fig. 4.5**(c)** some refinement was done near the non homogeneous Dirichlet boundaries.

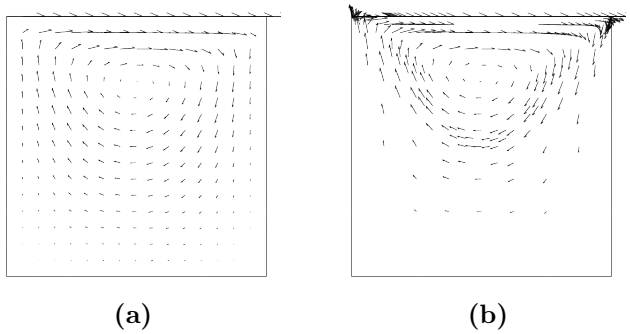


FIG. 4.2. *Velocity field on the uniform grid* **(a)** *and on the adapted grid* **(b)**.

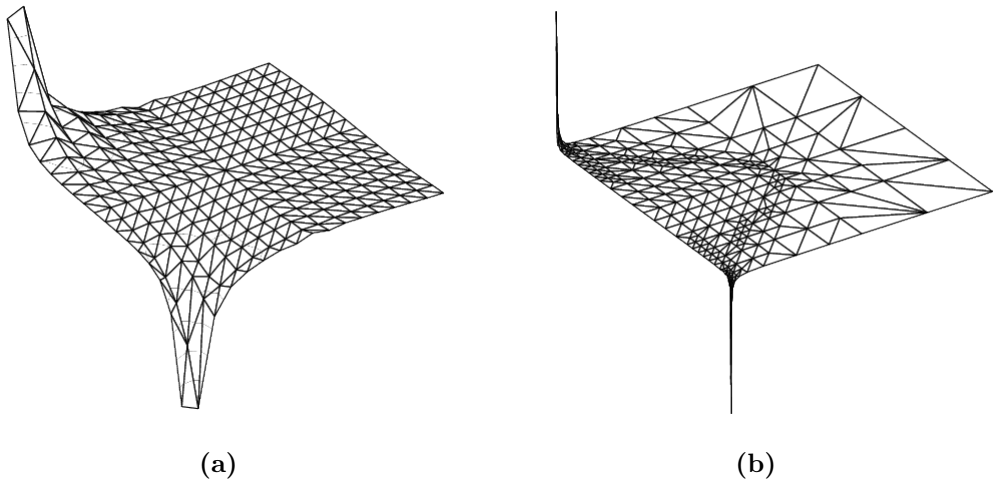
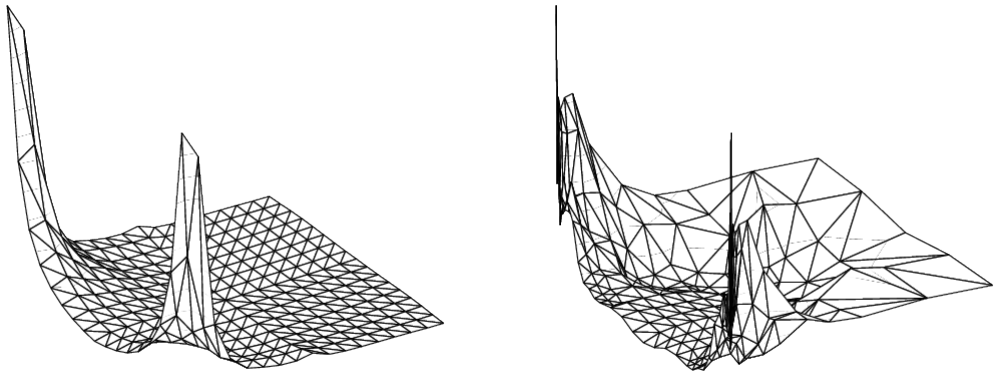
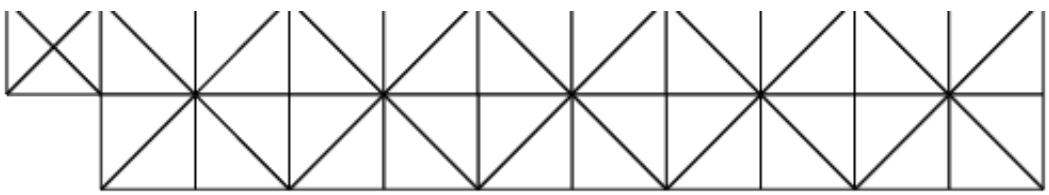


FIG. 4.3. *Pressure elevation on the uniform grid* **(a)** *and on the adapted grid* **(b)***; it is negative at the left upper corner of the cavity (suction effect) and positive at the right upper corner; at these points the level of the pressure is actually much higher on the adapted grid (order of magnitude* 10^2*) than on the uniform mesh (order of magnitude* 10^0*).*

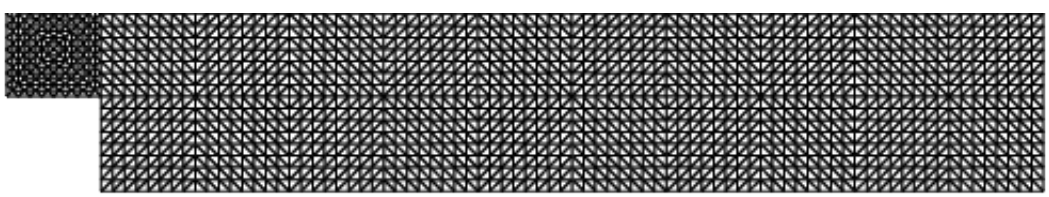


(a) **(b)**

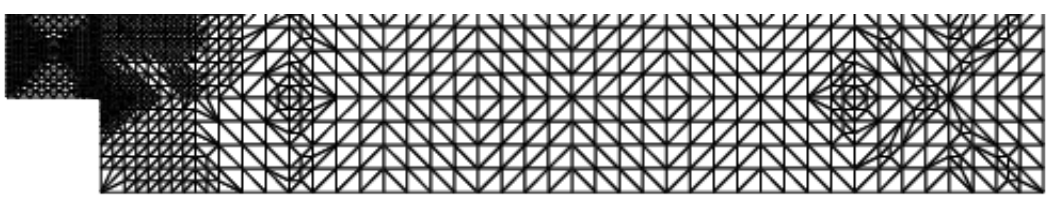
FIG. 4.4. *Estimated error on the uniform* **(a)** *and adapted* **(b)** *grids. Although the representation* **(b)** *is less smooth than* **(a)**, *the errors are much more uniformly reparted in* **(b)** *than in* **(a)**.



(a) *Initial grid with* $NV = 36$ *vertices and* $NT = 44$ *triangles.*

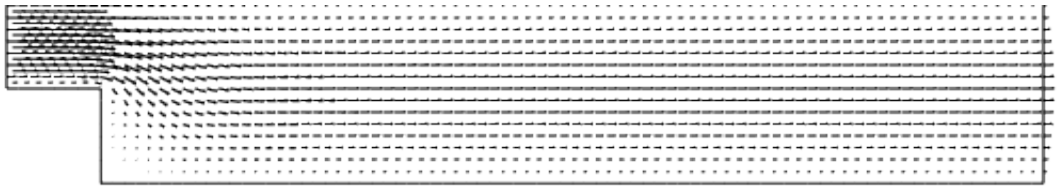


(b) *Uniformly refined grid with* $NV = 1513$ *vertices and* $NT = 2816$ *triangles (4 levels).*

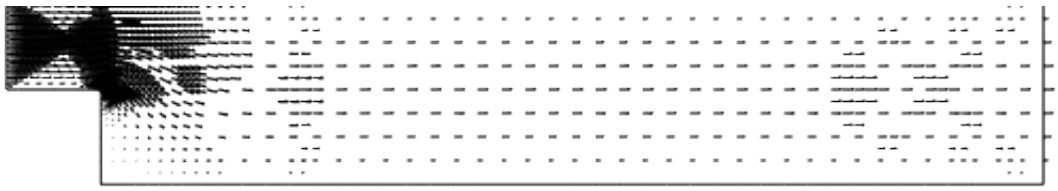


(c) *Adaptively refined grid with* $NV = 1517$ *vertices and* $NT = 2863$ *triangles (9 levels).*

FIG. 4.5.



(a) *Velocity field on the uniform grid.*



(b) *Velocity field on the adapted grid.*

FIG. 4.6.

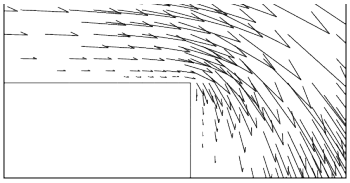
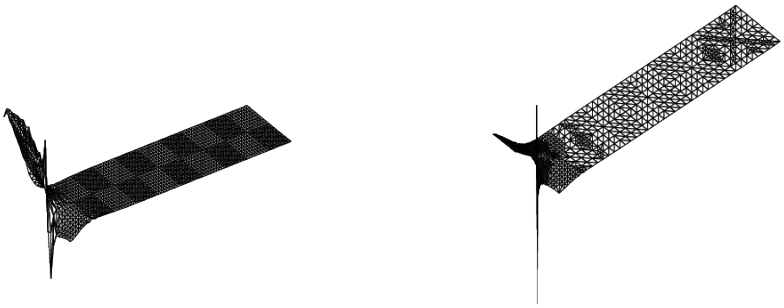
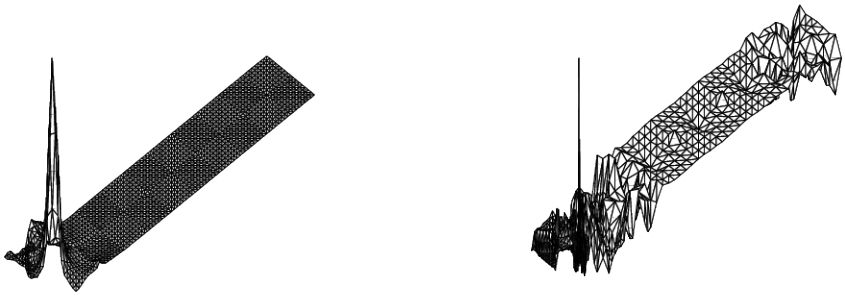


FIG. 4.7. *Detail of velocity field near the re-entrant corner of the step.*



(a) **(b)**

FIG. 4.8. *Pressure elevation on the uniform grid* **(a)** *and on the adapted grid* **(b)***; note that at the top corner of the step level of the pressure is higher on the adapted grid (order of magnitude* 10^2*) than on the uniform mesh (order of magnitude* 10^1*).*



(a) **(b)**

FIG. 4.9. *Estimated error on the uniform* **(a)** *and adapted* **(b)** *grids.*

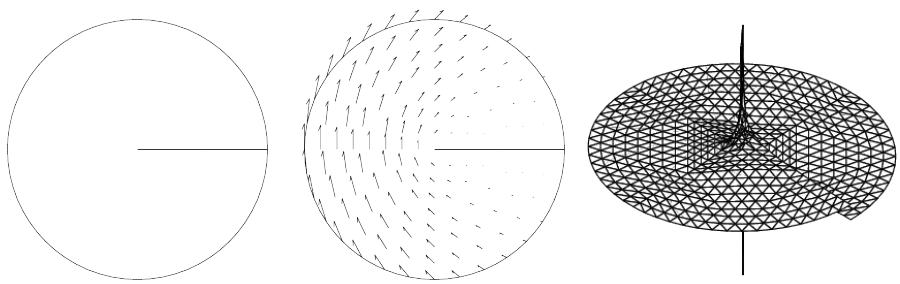
4.3. Disk with a Crack. Finally we test our error estimate on a Stokes flow in a disk of radius 1 with a crack joining the center to the boundary; the right-hand side $\boldsymbol{f}$ is 0 and the boundary conditions are

$$\boldsymbol{u}^t = \frac{3}{2}\left(\cos\frac{\theta}{2} - \cos\frac{3\theta}{2},\ 3\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right)$$

where (r, θ) is a polar representation of a point in the disk. The exact solution is then given by

$$\boldsymbol{u}^t = \frac{3\sqrt{r}}{2}\left(\cos\frac{\theta}{2} - \cos\frac{3\theta}{2},\ 3\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right)$$
$$p = -\frac{6}{\sqrt{r}}\ \cos\frac{\theta}{2}$$

and is singular at the end of the crack, i.e. at the center of the disk.



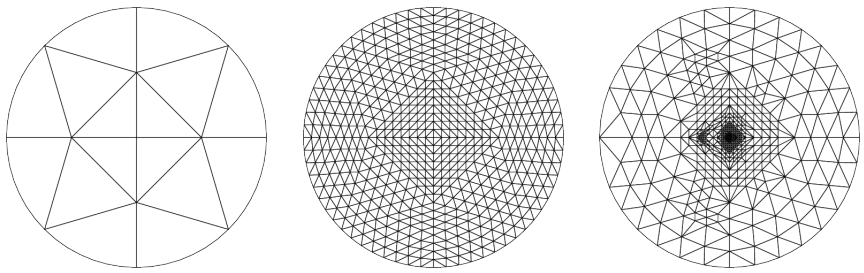
(a) **(b)** **(c)**

FIG. 4.10. **(a)** *Domain ;* **(b)** *velocity ;* **(c)** *pressure.*

We first solve our Stokes problem on a coarse grid consisting of $NV = 15$ vertices and $NT = 16$ triangles (Fig. 4.10**(a)**), then refine either uniformly or adaptively, thus creating two sequences of meshes of increasing and comparable size (or degrees of freedom = d. of f.)(see Tables 4.1 and 4.2) ($\nu = 1$).

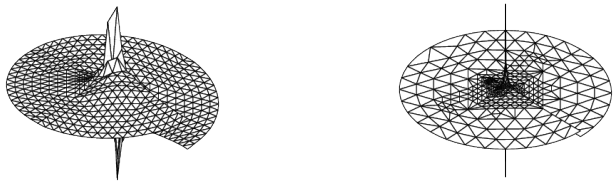
Pressure elevations are plotted on the uniform and adaptive refined grids (Figures 4.12**(a)(b)**).

The solution was computed on each grid, along with error estimates. During the refinement process, these estimates were computed using an interpolation scheme (one Jacobi sweep) for the values at the new nodes. Consequently their accuracy deteriorates along with the number of refinement steps; however had we used intermediate recalculations to base the computation of the estimates on, we would have gotten a mesh with more levels of refinement around the singularity (hence giving a higher level for the pressure). in that regard the use of interpolated values instead of computed solution values had a grid smoothing effect.



(a) **(b)** **(c)**

FIG. 4.11. **(a)** *Initial grid with* $NV = 15$ *vertices and* $NT = 16$ *triangles;* **(b)** *uniform mesh with* $NV = 561$ *vertices and* $NT = 1024$ *triangles (4 levels) and* **(c)** *adapted mesh with* $NV = 563$ *vertices and* $NT = 1054$ *triangles (9 levels).*



(a) **(b)**

FIG. 4.12. *Pressure elevation on the uniform* **(a)** *and adapted* **(b)** *meshes. The level of pressure is much higher on the adapted grid (order of magnitude* 10^3*) than on the uniform mesh (order of magnitude* 10^2*).*

We can measure the efficiency of our estimate both locally and globally:

- we perform a convergence analysis on the two sequences of (uniform and adapted) meshes and evaluate in particular the "effectivity ratio" q defined as the ratio of the exact global error $|||(\boldsymbol{e}, \epsilon)|||$ to the estimated global error $|||(\boldsymbol{e}^{''}, \epsilon^{''})|||$, and report the results in Tables 4.1 and 4.2. The column labeled "digits" gives the estimated number of correct digits in the solution of the discrete problem.

Uniform refinement					
levels	NV	NT	d. of f.	digits	q
1	15	16	53	0.341	0.74
2	45	64	215	0.455	0.99
3	153	256	875	0.586	1.10
4	561	1024	3539	0.732	1.13
5	2145	4096	14243	0.883	1.14

TABLE 4.1
Convergence analysis and effectivity ratio for a sequence of uniform meshes.

Adaptive refinement					
levels	NV	NT	d. of f.	digits	q
1	15	16	53	0.341	0.74
3	41	64	219	0.606	0.50
5	143	250	861	0.772	0.73
8	556	1042	3616	1.077	0.93
12	2139	4124	14361	1.413	1.11

TABLE 4.2
Convergence analysis and effectivity ratio for a sequence of adapted meshes.

  From the results in the column "digits" it is clear that the convergence is much better in the adaptive case than in the uniform one.

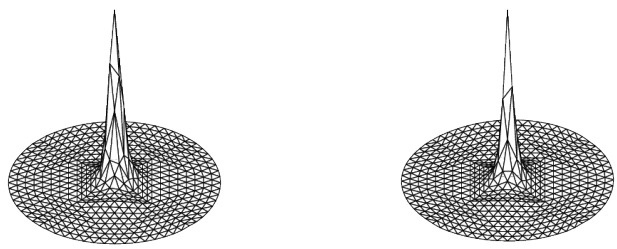
- we plot the estimated error vs the exact error to test the local behavior of the estimate, and its propensity to recognize the regions in the mesh needing some refinement or unrefinement (Figures 4.13**(a)(b)** and 4.14**(a)(b)**).

In Table 4.3 are listed the convergence rates γ for both uniform and adaptive cases, in the energy norm, as well as in the $\mathcal{H}_0^1$ norm ($\|\nabla \cdot \|$) for the velocity and the $\mathcal{L}^2$ norm of the pressure, which both are regrouped in the energy norm. These numbers are such that $10^{-digits} \sim NV^{-\gamma/2} \sim h^{\gamma}$ (in the least square sense). The row labeled "B-D" refers to *a priori* estimates results published by Brezzi and Douglas [8].

All convergence rates are based on a least square fitting from the number of correct digits in the solution for each of the grids in the (uniform and adapted) sequences. In the uniform case these values are smaller than expected from the a priori estimates given by Brezzi and Douglas, except maybe for the $\mathcal{H}_0^1$ norm of the pressure, mainly because the solution does not have here the regularity required in the derivation of their estimates. Note the singularity has about the same effect on the $\mathcal{L}^2$ and $\mathcal{H}_0^1$ norms for the velocity, this effect being less obvious on the pressure. Note also that the difference between the convergence in $\mathcal{L}^2$ and $\mathcal{H}_0^1$ norms is of the order of unity,

which corresponds to one power of h in the estimates in [8]. In the adaptive strategy the rates of convergence are increased back to the expected levels (velocity) or more (pressure), yielding a superlinear convergence in the energy norm (on a computational point of view, the adaptive refinement has somehow regularized the solution around the singularity).

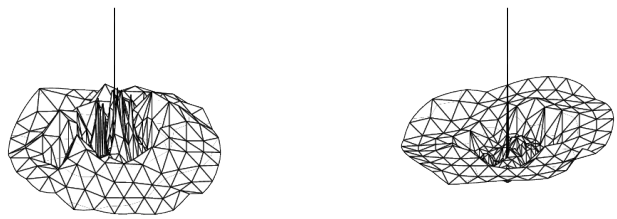
Finally we plot the (estimated and exact) errors in both cases. Since the estimates are given by triangles, they are transformed into errors on the vertices by averaging between all the triangles neighbor of a node, as in the previous examples. On the other hand exact errors are known at the vertices. However, in order to compare similar results, these are used to compute errors in each triangle based on the energy norm of interpolated errors at the midpoints of all interior edges. Then an averaging identical to the one above is performed to get an error at the vertices. On Figures 4.13**(a)(b)** (uniform case) and 4.14**(a)(b)** (adapted case) we can note that the estimate is in good agreement with the exact error, thus providing a nice tool for adapting grids, especially when discontinuities or steep variations in the solution occur.



(a) **(b)**

FIG. 4.13. **(a)** *Estimated error and* **(b)** *exact error on the uniform grid. The error ranges from* 7.10^{-3} *to* $8\ 10^{-1}$ *for both estimated and exact errors.*

5. Conclusion. In this paper we derived an a posteriori error estimate which can be used toward the solution of the Stokes equations on geometries of industrial interest. On the test problems of section 4 the computation of this estimate took about one fourth of the total time for the solution process. It remains to compare it with other estimates, in particular with those derived by R. Verfürth in [16].



(a) **(b)**

FIG. 4.14. **(a)** *Estimated error and* **(b)** *exact error on the adapted grid. The estimated error ranges from* 2.10^{-3} *to* 10^{-1}*, compared to* 3.10^{-3} *to* 2.10^{-1} *for the exact error.*

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type of refinement	$\mathcal{L}^2$ norm (velocity)	$\mathcal{H}_0^1$ norm (velocity)	$\mathcal{L}^2$ norm (pressure)	$\mathcal{H}_0^1$ norm (pressure)	energy norm $\|\|\|(\cdot,\cdot)\|\|\|$
uniform	1.51	0.59	0.96	0.09	0.87
adaptive	1.79	1.03	1.66	0.29	1.46
B-D	2.00	1.00	1.00	0.00	1.00

TABLE 4.3
Rates of convergence in $\mathcal{L}^2, \mathcal{H}_0^1$ and energy norms.