Math 175/275: Numerical Methods for PDE

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Winter Quarter 2020
Final Examination
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**Question 1.** (Galerkin’s Method.) Let \( a(\cdot, \cdot) \) and \( L(\cdot) \) satisfy the assumptions of the Lax-Milgram lemma, i.e.,

\[
|a(v, w)| \leq C_1 |v|_V |w|_V, \quad \forall v, w \in V \\
a(v, v) \geq C_2 |v|^2_V, \quad \forall v \in V \\
|L(v)| \leq C_3 |v|_V \\
\forall v \in V.
\]

Let \( u \) be the solution of

\[
a(u, v) = L(v), \quad \forall v \in V.
\]

Let \( \tilde{V} \subset V \) be a finite-dimensional subspace and let \( \tilde{u} \in \tilde{V} \) be determined by Galerkin’s method:

\[
a(\tilde{u}, v) = L(v), \quad \forall v \in \tilde{V}.
\]

Prove that (note that \( a(\cdot, \cdot) \) may not be symmetric)

\[
\|\tilde{u} - u\|_V \leq C_1 C_2 \min_{\chi \in \tilde{V}} \|\chi - u\|_V.
\]

Prove that if \( a(\cdot, \cdot) \) is symmetric and \( \|v\|_a = a(v, v)^{1/2} \), then

\[
\|\tilde{u} - u\|_a = \min_{\chi \in \tilde{V}} |\chi - u|_a.
\]

and

\[
\|\tilde{u} - u\|_V \leq \sqrt{\frac{C_1}{C_2}} \min_{\chi \in \tilde{V}} |\chi - u|_V.
\]
Question 2. Consider the problem

\[ u_t - \Delta u = f, \quad \text{in } \Omega \times R_+, \]
\[ u = 0, \quad \text{on } \Gamma \times R_+, \]
\[ u(\cdot, 0) = v, \quad \text{in } \Omega, \]

in the case of one space dimension with \( \Omega = (0, 1) \). For the numerical solution, we use piecewise linear functions based on the partition

\[ 0 < x_1 < x_2 < \cdots < x_M < 1, \quad x_j = jh, h = 1/(M + 1). \]

Determine the mass matrix \( B \) and the stiffness matrix \( A \) and write down the semidiscrete problem, the backward Euler equations, and the Crank-Nicolson equations.
Question 3. Consider the scalar first order differential equation

$$
\sum_{j=1}^{d} a_j(x) \frac{\partial u}{\partial x_j} + a_0(x)u \equiv a \cdot \nabla u + a_0 u = f(x), \quad x \in \Omega
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^d$ with boundary $\Gamma$, $a$ is a smooth vector field that does not vanish at any point, and $a_0$ and $f$ are given smooth functions. The solution is given on the inflow boundary

$$
u = v, \quad \text{on } \Gamma_-,
$$

where $\Gamma = \Gamma_+ \cup \Gamma_-$,

$$
\Gamma_- = \{x \in \Gamma : n(x) \cdot a(x) < 0\}, \\
\Gamma_+ = \{x \in \Gamma : n(x) \cdot a(x) \geq 0\},
$$

and $n$ is the exterior normal. Prove the following stability estimate for the above problem, under a suitable condition on the coefficients $a_j$:

$$
\int_{\Omega} u^2 \, dx + \int_{\Gamma_+} u^2 n \cdot a \, ds \leq C \left( \int_{\Omega} f^2 \, dx + \int_{\Gamma_-} v^2 |n \cdot a| \, ds \right).
$$