MATH 175/275: Numerical Methods for PDE

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Winter Quarter 2020
Final Examination
Monday, March 16, 2020

NAME ______________________

SIGNATURE ______________________

#1	30	
#2	40	
#3	30	
Total	100	

Question 1. (Galerkin's Method.) Let $a(\cdot,\cdot)$ and $L(\cdot)$ satisfy the assumptions of the Lax-Milgram lemma, i.e.,

$$\begin{aligned} |a(v,w)| &\le C_1\|v\|_V\|w\|_V, && \forall v,w\in V\\ a(v,v) &\ge C_2\|v\|_V^2 && \forall v\in V,\\ |L(v)| &\le C_3\|v\|_V && \forall v\in V. \end{aligned}$$

Let u be the solution of

$$a(u,v)=L(v), \qquad \forall v\in V.$$

Let $\tilde V\subset V$ be a finite-dimensional subspace and let $\tilde u\in\tilde V$ be determined by Galerkin's method:

$$a(\tilde u,v)=L(v), \qquad \forall v\in \tilde V.$$

Prove that (note that $a(\cdot,\cdot)$ may not be symmetric)

$$\|\tilde u-u\|_V\le \frac{C_1}{C_2}\min_{\chi\in\tilde V}\|\chi-u\|_V.$$

Prove that if $a(\cdot,\cdot)$ is symmetric and $\|v\|_a=a(v,v)^{1/2}$, then

$$\|\tilde u-u\|_a=\min_{\chi\in\tilde V}\|\chi-u\|_a.$$

and

$$\|\tilde u-u\|_V\le \sqrt{\frac{C_1}{C_2}}\min_{\chi\in\tilde V}\|\chi-u\|_V.$$

Question 2. Consider the problem

$$\begin{aligned} u_t - \Delta u &= f, && \text{in } \Omega \times R_+, \\ u &= 0, && \text{on } \Gamma \times R_+, \\ u(\cdot, 0) &= v, && \text{in } \Omega, \end{aligned}$$

in the case of one space dimension with $\Omega = (0, 1)$. For the numerical solution, we use piecewise linear functions based on the partition

$$0 < x_1 < x_2 < \cdots < x_M < 1, \qquad x_j = jh, h = 1/(M+1).$$

Determine the mass matrix B and the stiffness matrix A and write down the semidiscrete problem, the backward Euler equations, and the Crank-Nicolson equations.

Question 3. Consider the scalar first order differential equation

$$\sum_{j=1}^{d} a_j(x) \frac{\partial u}{\partial x_j} + a_0(x)u \equiv a \cdot \nabla u + a_0 u = f(x), \qquad x \in \Omega$$

where Ω is a bounded domain in $\mathcal{R}^d$ with boundary Γ, a is a smooth vector field that does not vanish at any point, and a_0 and f are given smooth functions. The solution is given on the inflow boundary

$$u = v, \qquad \text{on } \Gamma_-,$$

where $\Gamma \equiv \Gamma_+ \cup \Gamma_-$,

$$\Gamma_- = \{x \in \Gamma : n(x) \cdot a(x) < 0\},$$
$$\Gamma_+ = \{x \in \Gamma : n(x) \cdot a(x) \geq 0\},$$

and n is the exterior normal. Prove the following stability estimate for the above problem, under a suitable condition on the coefficients a_j:

$$\int_\Omega u^2 \, dx + \int_{\Gamma_+} u^2 n \cdot a \, ds \leq C \left(\int_\Omega f^2 \, dx + \int_{\Gamma_-} v^2 |n \cdot a| \, ds \right).$$