Exercise 8.1. This problem is about the relationship between the Ritz and Galerkin methods. Let $\mathcal{H}$ be a space of functions and consider the Ritz minimization problem: find $u^* \in \mathcal{H}$ which satisfies

$$\frac{a(u^*, u^*)}{2} - (f, u^*) = \min_{v \in \mathcal{H}} \frac{a(v, v)}{2} - (f, v)$$

(8.1)

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad (f, v) = \int_\Omega fv \, dx$$

(this is a weak form of the problem: $-\Delta u = f$ in $\Omega$, $u = 0$ on $\partial \Omega$.) Show that the Ritz formulation (8.1) is equivalent to the Galerkin formulation: find $u^* \in \mathcal{H}$ such that

$$a(u^*, v) = (f, v)$$

for all $v \in \mathcal{H}$. (hint: try a function of the form $u^* + \epsilon v$ in $a(\cdot, \cdot)/2 - (f, \cdot)$, where $u^* \in \mathcal{H}$ is the solution of (8.1), $v \in \mathcal{H}$ and $\epsilon$ is a scalar).

Exercise 8.2. Let $S_h \subset H$ be and suppose that $u^* \in \mathcal{H}$ solves the minimization problem

$$\frac{a(u^*, u^*)}{2} - (f, u^*) = \min_{v \in \mathcal{H}} \frac{a(v, v)}{2} - (f, v)$$

(8.2)

and $u^*_h \in S_h$ solves the minimization problem

$$\frac{a(u^*_h, u^*_h)}{2} - (f, u^*_h) = \min_{v \in S_h} \frac{a(v, v)}{2} - (f, v)$$

(8.3)

with $a(\cdot, \cdot)$ and $(\cdot, \cdot)$ defined as in the Question 1. Prove

$$\|u^* - u^*_h\| = \min_{v \in S_h} \|u^* - v\|$$

where $\|v\|^2 = a(v, v)$. (Hint: apply Question 1 to both $u^*$ and $u^*_h$ to first show that $a(u^* - u^*_h, v) = 0$ for all $v \in S_h$.)
