# Math 270c: Numerical Ordinary Differential Equations

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Spring Quarter 2018

Homework Assignment #8
Due Friday, June 8, 2018

**Exercise 8.1.** This problem is about the relationship between the Ritz and Galerkin methods. Let $\mathcal{H}$ be a space of functions and consider the *Ritz* minimization problem: find $u^* \in \mathcal{H}$ which satisfies

$$\frac{a(u^*, u^*)}{2} - (f, u^*) = \min_{v \in \mathcal{H}} \frac{a(v, v)}{2} - (f, v) \tag{8.1}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad (f, v) = \int_\Omega f v \, dx$$

(this is a weak form of the problem:$-\Delta u = f$ in $\Omega$, $u = 0$ on $\partial\Omega$.) Show that the Ritz formulation (8.1) is equivalent to the *Galerkin* formulation: find $u^* \in \mathcal{H}$ such that

$$a(u^*, v) = (f, v)$$

for all $v \in \mathcal{H}$. (hint: try a function of the form $u^* + \epsilon v$ in $a(\cdot, \cdot)/2 - (f, \cdot)$, where $u^* \in \mathcal{H}$ is the solution of (8.1), $v \in \mathcal{H}$ and $\epsilon$ is a scalar).

**Exercise 8.2.** Let $\mathcal{S}_h \subset H$ be and suppose that $u^* \in \mathcal{H}$ solves the minimization problem

$$\frac{a(u^*, u^*)}{2} - (f, u^*) = \min_{v \in \mathcal{H}} \frac{a(v, v)}{2} - (f, v) \tag{8.2}$$

and $u_h^* \in \mathcal{S}_h$ solves the minimization problem

$$\frac{a(u_h^*, u_h^*)}{2} - (f, u_h^*) = \min_{v \in \mathcal{S}_h} \frac{a(v, v)}{2} - (f, v) \tag{8.3}$$

with $a(\cdot, \cdot)$ and $(\cdot, \cdot)$ defined as in the Question 1. Prove

$$|||u^* - u_h^*||| = \min_{v \in \mathcal{S}_h} |||u^* - v|||$$

where $|||v|||^2 = a(v, v)$. (Hint: apply Question 1 to both $u^*$ and $u_h^*$ to first show that $a(u^* - u_h^*, v) = 0$ for all $v \in \mathcal{S}_h$.