Math 270c: Numerical Ordinary Differential Equations

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Spring Quarter 2018

Homework Assignment #7
Due Friday, June 1, 2018

Exercise 7.1. Let $x_0 = x_{n+1} = 0$, $h = 1/n$, and define $\|x\|^2 = h\sum_{i=1}^n x_i^2$. Prove the discrete Poincare inequality

$$h\sum_{i=1}^{n} x_i^2 \le Ch\sum_{i=1}^{n+1}\left(\frac{x_i - x_{i-1}}{h}\right)^2$$

Exercise 7.2. Let A be the $n \times n$ symmetric positive definite tridiagonal matrix with elements $a_{i,i} = (p_{i-1/2} + p_{i+1/2})/h^2 + q_i$ and $a_{i,i+1} = -p_{i+1/2}/h^2$ with $p_{i+1/2} > 0$ and $q_i \ge 0$. Let B be the $n \times n$ symmetric positive definite tridiagonal matrix constant diagonal elements $2/h^2$ and off-diagonal elements $-1/h^2$. Prove there exist positive constants c_1 and c_2, depending only on the $p_{i+1/2}$ and q_i, such that

$$c_1 \le \frac{x^t A x}{x^t B x} \le c_2.$$

(Use exercise 7.1 and use $x_0 = x_{n+1} = 0$.)

Exercise 7.3. Consider the two point boundary value problem

$$-u'' + \gamma u = f$$

for $0 < x < 1$, with $u(0) = u(1) = 0$, and $\gamma \ge 0$ a scalar constant. Find a *compact* fourth order discretization of the form

$$\frac{-u(x+h) + 2u(x) - u(x-h)}{h^2} + \gamma\{A(u(x+h) + u(x-h)) + (1-2A)u(x)\}$$
$$= B(f(x+h) + f(x-h)) + (1-2B)f(x)$$

where A and B are constants.

Exercise 7.4. Suppose

$$a_i x_i - b_i x_{i+1} - b_{i-1} x_{i-1} = 0$$

for $1 \le i \le n$, with x_0 and x_{n+1} given. Assume $b_i > 0$, $0 \le i \le n$ and $a_i = b_i + b_{i-1}$, $1 \le i \le n$. Prove the *Discrete Maximum Principle*

$$\max\{x_0, x_{n+1}\} = \max_{0 \le i \le n+1} x_i$$