Math 270c: Numerical Ordinary Differential Equations

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Homework Assignment #7
Due Friday, June 1, 2018

Exercise 7.1. Let \( x_0 = 0 \), \( h = 1/n \), and define \( ||x||^2 = h \sum_{i=1}^{n} x_i^2 \). Prove the discrete Poincare inequality
\[
   h \sum_{i=1}^{n} x_i^2 \leq Ch \sum_{i=1}^{n} \left( \frac{x_i - x_{i-1}}{h} \right)^2
\]

Exercise 7.2. Let \( A \) be the \( n \times n \) symmetric positive definite tridiagonal matrix with elements \( a_{i,i} = (p_{i-1/2} + p_{i+1/2})/h^2 + q_i \) and \( a_{i,i+1} = -p_{i+1/2}/h^2 \) with \( p_{i+1/2} > 0 \) and \( q_i \geq 0 \). Let \( B \) be the \( n \times n \) symmetric positive definite tridiagonal matrix constant diagonal elements \( 2/h^2 \) and off-diagonal elements \( -1/h^2 \). Prove there exist positive constants \( c_1 \) and \( c_2 \), depending only on the \( p_{i+1/2} \) and \( q_i \), such that
\[
   c_1 \leq \frac{x^t A x}{x^t B x} \leq c_2.
\]
(Use exercise 7.1)

Exercise 7.3. Consider the two point boundary value problem
\[
   -u'' + \gamma u = f
\]
for \( 0 < x < 1 \), with \( u(0) = u(1) = 0 \), and \( \gamma \geq 0 \) a scalar constant. Find a compact fourth order discretization of the form
\[
   -\frac{u(x + h) + 2u(x) - u(x - h)}{h^2} + \gamma \{u(x + h) + u(x - h) + (1 - 2A)u(x)\} = B(f(x + h) + f(x - h) + (1 - 2B)f(x)
\]
where \( A \) and \( B \) are constants.

Exercise 7.4. Suppose
\[
   a_i x_i - b_i x_{i+1} - b_{i-1} x_{i-1} = 0
\]
for \( 1 \leq i \leq n \), with \( x_0 \) and \( x_{n+1} \) given. Assume \( b_i > 0 \), \( 0 \leq i \leq n \) and \( a_i = b_i + b_{i-1} \), \( 1 \leq i \leq n \). Prove the Discrete Maximum Principle
\[
   \max\{x_0, x_{n+1}\} = \max_{0 \leq i \leq n+1} x_i
\]