Math 270c: Numerical Ordinary Differential Equations

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Spring Quarter 2018

Homework Assignment #6
Due Friday, May 18, 2018

Exercise 6.1. Let

$$Lu = -(pu')'$$

for $0 \leq x \leq 1$, where $p(x)$ is smooth and $p(x) \geq p_{\min} > 0$. In this problem we will prove the *maximum principle*: if $u \in \mathcal{C}^2(0,1)$ and $Lu \leq 0$ for $0 \leq x \leq 1$, then

$$\max_{0 \leq x \leq 1} u(x) = \max\{u(0), u(1)\}.$$

We will prove this result in several steps.

a. First assume $Lu < 0$ and assume that the maximum occurs at x_0 where $0 < x_0 < 1$. Since $u'(x_0) = 0$, $u''(x_0) \leq 0$, show this leads to a contradiction.

b. Now assume $Lu \leq 0$. Let $\phi = e^{\lambda x}$ with λ sufficiently large that $L\phi < 0$ for $0 \leq x \leq 1$. Again assume the maximum occurs at x_0. Use the function $v = u + \epsilon\phi$ for $\epsilon > 0$ but sufficiently small, and apply part a.

Exercise 6.2. Now let

$$Lu = -(pu')' + qu$$

for $0 \leq x \leq 1$, where $p(x)$ are $q(x)$ are smooth, $p(x) \geq p_{\min} > 0$, and $q(x) \geq 0$. In this problem we will prove the *maximum principle*: if $u \in \mathcal{C}^2(0,1)$ and $Lu \leq 0$ for $0 \leq x \leq 1$, then

$$\max_{0 \leq x \leq 1} u(x) \leq \max\{u(0), u(1), 0\}.$$

If $u \leq 0$, we are done. As before assume the maximum occurs at x_0 and $u(x_0) > 0$. Let (α, β) be the biggest interval on $(0,1)$ where $u > 0$, and consider the operator $\hat{L}u = Lu - qu$, and use the previous exercise to show a contradiction.