Math 270b: Numerical Approximation and Nonlinear Equations

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Winter Quarter 2020

Homework Assignment #9
Due Friday, March 6, 2020

Exercise 9.1. Let

$$\mathcal{I}(f) = \int_0^1 f(x)dx$$

We consider a numerical quadrature formula of the form

$$\mathcal{Q}(f) = w_1 f(1/3) + w_2 f(2/3) + w_3 f(1)$$

1. Compute the weights w_i to maximize the order.
2. Compute the error $\mathcal{I}(f) - \mathcal{Q}(f)$. Be sure to explicitly evaluate the constant.
3. Write down the composite formula for approximating

   $$\int_a^b f(x)dx$$

   on a uniform mesh of size h.
4. Write down an expression for the error in the composite formula.

Exercise 9.2. Derive the Peano kernel for the midpoint rule:

$$\int_0^1 f(x)\,dx - f(1/2) = \int_0^1 K(t) f''(t)\,dt$$

where

$$K(t) = \begin{cases} t^2/2 & 0 \le t \le 1/2 \\ (1-t)^2/2 & 1/2 \le t \le 1 \end{cases}$$

Exercise 9.3. Let

$$I(f) = \int_0^1 f(x) \log\left(\frac{1}{x}\right) dx$$

1. Derive the two point Gaussian quadrature rule for approximating this integral, where the weight function is $w(x) = \log\left(\frac{1}{x}\right)$
2. Determine the order of this formula.

3. Suppose we wish to compute

$$\int_0^a f(x) \log\left(\frac{1}{x}\right)\, dx = \int_0^h f(x) \log\left(\frac{1}{x}\right)\, dx + \int_h^a f(x) \log\left(\frac{1}{x}\right)\, dx$$

   Show how the special rule derived above could be combined with a standard rule (e.g. 2-point Gaussian Quadrature with weight function 1) to make a composite formula.

Exercise 9.4. Suppose we have a numerical procedure for approximating a number $\mathcal{X}$ by $\mathcal{X}_h$. It is (somehow) known that the approximation error is given by an asymptotic expansion of the form

$$\mathcal{X} - \mathcal{X}_h = c_1 h + c_2 h^2 + \cdots + c_k h^k \ldots$$

where h is a small parameter characterizing the discretization. Devise a Richardson Extrapolation scheme for computing higher order approximations of $\mathcal{X}$ given first order approximations for $h_j = h_0/3^j$, $j = 0, 1, 2, \ldots$.