Exercise 9.1. Let

\[ I(f) = \int_0^1 f(x) \, dx \]

We consider a numerical quadrature formula of the form

\[ Q(f) = w_1 f(1/3) + w_2 f(2/3) + w_3 f(1) \]

1. Compute the weights \( w_i \) to maximize the order.
2. Compute the error \( I(f) - Q(f) \). Be sure to explicitly evaluate the constant.
3. Write down the composite formula for approximating

\[ \int_a^b f(x) \, dx \]

on a uniform mesh of size \( h \).
4. Write down an expression for the error in the composite formula.

Exercise 9.2. Derive the Peano kernel for the midpoint rule:

\[ \int_0^1 f(x) \, dx - f(1/2) = \int_0^1 K(t) f''(t) \, dt \]

where

\[ K(t) = \begin{cases} \frac{t^2}{2} & 0 \leq t \leq 1/2 \\ \frac{(1-t)^2}{2} & 1/2 \leq t \leq 1 \end{cases} \]

Exercise 9.3. Let

\[ I(f) = \int_0^1 f(x) \log \left( \frac{1}{x} \right) \, dx \]

1. Derive the two point Gaussian quadrature rule for approximating this integral, where the weight function is \( w(x) = \log \left( \frac{1}{x} \right) \)
2. Determine the order of this formula.
3. Suppose we wish to compute

\[ \int_0^a f(x) \log \left( \frac{1}{x} \right) \, dx = \int_0^h f(x) \log \left( \frac{1}{x} \right) \, dx + \int_h^a f(x) \log \left( \frac{1}{x} \right) \, dx \]

Show how the special rule derived above could be combined with a standard rule (e.g. 2-point Gaussian Quadrature with weight function 1) to make a composite formula.

**Exercise 9.4.** Suppose we have a numerical procedure for approximating a number \( X \) by \( X_h \). It is (somehow) known that the approximation error is given by an asymptotic expansion of the form

\[ X - X_h = c_1 h + c_2 h^2 + \cdots + c_k h^k \cdots \]

where \( h \) is a small parameter characterizing the discretization. Devise a Richardson Extrapolation scheme for computing higher order approximations of \( X \) given first order approximations for \( h_j = h_0/3^j, j = 0, 1, 2, \ldots \).