Math 270b: Numerical Approximation and Nonlinear Equations

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Winter Quarter 2020

Homework Assignment #7
Due Friday, February 21, 2020

Exercise 7.1. Suppose we wish the approximate the function $f(x) = e^x$ on the interval $0 \le x \le 1$ using piecewise polynomial approximation on a uniform mesh of $n+1$ knots $(h = 1/n)$.

1. If we use $\mathcal{C}^0$ piecewise linear interpolation, compute the smallest value of n such that

$$\|f - \mathcal{I}_1(f)\|_\infty \le 10^{-6}$$

2. If we use $\mathcal{C}^1$ piecewise cubic interpolation, compute the smallest value of n such that

$$\|f - \mathcal{I}_3(f)\|_\infty \le 10^{-6}$$

3. Give a brief comparison of the relative efficiency of these two methods.

Exercise 7.2. In this problem we will analyze the case of continuous piecewise *quadratic* interpolation on a mesh of $n+1$ knots $x_0 < x_1 < \cdots < x_n$. We will also need the interval midpoints $x_{i+1/2} = (x_i + x_{i+1})/2$.

1. Show the the dimension of the space $\mathcal{S}$ of continuous piecewise quadratic polynomials is $N = 2n+1$.
2. Next compute the *nodal* basis functions. There are two types: *hat functions*, which satisfy

$$\phi_i(x_j) = \delta_{ij} \qquad \phi_i(x_{j+1/2}) = 0 \qquad 0 \le i \le n,$$

and *bump functions*, which satisfy

$$\phi_{i+1/2}(x_j) = 0 \qquad \phi_{i+1/2}(x_{j+1/2}) = \delta_{ij} \qquad 0 \le i \le n-1.$$

Draw a picture of both types of basis functions.

3. Let f^* be the continuous piecewise quadratic interpolant for f. Prove

$$\begin{aligned} \|f - f^*\|_\infty &\le Ch^3\|f'''\|_\infty \\ \|f' - f^{*\prime}\|_\infty &\le Ch^2\|f'''\|_\infty \end{aligned}$$

4. Prove

$$\begin{aligned} \|f - f^*\|_2 &\le Ch^3\|f'''\|_2 \\ \|f' - f^{*\prime}\|_2 &\le Ch^2\|f'''\|_2 \end{aligned}$$

Exercise 7.3. In this problem we will analyze the case of $\mathcal{C}^1$ piecewise *quadratic* approximation on a mesh of $n+1$ knots $x_0 < x_1 < \cdots < x_n$.

1. Show the the dimension of the space $\mathcal{S}$ of $\mathcal{C}^1$ piecewise quadratic polynomials is $N = n+2$.
2. As with cubic splines, there are no simple nodal basic functions for this space. Show that the mininimum support for a quadratic spline basis function is three intervals.
3. Compute the quadratic spline basis functions.