Math 270B: Numerical Approximation and Nonlinear Equations

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Winter Quarter 2020

Homework Assignment #7
Due Friday, February 21, 2020

Exercise 7.1. Suppose we wish to approximate the function \( f(x) = e^x \) on the interval \( 0 \leq x \leq 1 \) using piecewise polynomial approximation on a uniform mesh of \( n + 1 \) knots \( (h = 1/n) \).

1. If we use \( C^0 \) piecewise linear interpolation, compute the smallest value of \( n \) such that
\[
\| f - I_1(f) \|_\infty \leq 10^{-6}
\]

2. If we use \( C^1 \) piecewise cubic interpolation, compute the smallest value of \( n \) such that
\[
\| f - I_3(f) \|_\infty \leq 10^{-6}
\]

3. Give a brief comparison of the relative efficiency of these two methods.

Exercise 7.2. In this problem we will analyze the case of continuous piecewise quadratic interpolation on a mesh of \( n + 1 \) knots \( x_0 < x_1 < \cdots < x_n \). We will also need the interval midpoints \( x_{i+1/2} = (x_i + x_{i+1})/2 \).

1. Show the dimension of the space \( \mathcal{S} \) of continuous piecewise quadratic polynomials is \( N = 2n + 1 \).

2. Next compute the nodal basis functions. There are two types: hat functions, which satisfy
\[
\phi_i(x_j) = \delta_{ij}, \quad \phi_i(x_{j+1/2}) = 0 \quad 0 \leq i \leq n,
\]
and bump functions, which satisfy
\[
\phi_{i+1/2}(x_j) = 0, \quad \phi_{i+1/2}(x_{j+1/2}) = \delta_{ij}, \quad 0 \leq i \leq n - 1.
\]

Draw a picture of both types of basis functions.

3. Let \( f^* \) be the continuous piecewise quadratic interpolant for \( f \). Prove
\[
\| f - f^* \|_\infty \leq C h^3 |f'''|_\infty
\]
\[
\| f' - f'^* \|_\infty \leq C h^2 |f'''|_\infty
\]

4. Prove
\[
\| f - f^* \|_2 \leq C h^3 |f'''|_2
\]
\[
\| f' - f'^* \|_2 \leq C h^2 |f'''|_2
\]
Exercise 7.3. In this problem we will analyze the case of $C^1$ piecewise quadratic approximation on a mesh of $n+1$ knots $x_0 < x_1 < \cdots < x_n$.

1. Show the dimension of the space $S$ of $C^1$ piecewise quadratic polynomials is $N = n + 2$.

2. As with cubic splines, there are no simple nodal basic functions for this space. Show that the minimum support for a quadratic spline basis function is three intervals.

3. Compute the quadratic spline basis functions.