MATH 270B: Numerical Approximation and Nonlinear Equations

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Winter Quarter 2020

Homework Assignment #6
Due Friday, February 14, 2020

Exercise 6.1. This is a big problem which establishes the basic theory for the case of Lagrange interpolation with knots of arbitary multiplicity.

1. Let x_i be a knot of multiplicity m_i, $0 \le i \le n$, with the x_i distinct. Then

$$f[x_0^{\langle m_0 \rangle}, x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}] = \frac{1}{\prod_{k=0}^{n}(m_k-1)!} \frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}} \cdots \frac{\partial^{m_n-1}}{\partial x_n^{m_n-1}} f[x_0, x_1, \ldots, x_n]$$

   Prove:

$$f[x_0^{\langle m_0 \rangle}, x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}] = \frac{1}{m_0-1} \frac{\partial}{\partial x_0} f[x_0^{\langle m_0-1 \rangle}, x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}]$$

2. Let $m_0 = m_n = 1$. Prove:

$$f[x_0, x_1^{\langle m_1 \rangle}, \ldots, x_{n-1}^{\langle m_{n-1} \rangle}, x_n] = \frac{f[x_0, x_1^{\langle m_1 \rangle}, \ldots, x_{n-1}^{\langle m_{n-1} \rangle}] - f[x_1^{\langle m_1 \rangle}, \ldots, x_{n-1}^{\langle m_{n-1} \rangle}, x_n]}{x_0 - x_n}$$

   Use direct substitution.

3. Let $m_0 = 1$ and $m_n \ge 1$. Prove:

$$f[x_0, x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}] = \frac{f[x_0, x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n-1 \rangle}] - f[x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}]}{x_0 - x_n}$$

   Prove this by induction on m_n, using parts 1 and 2.

4. Let $m_0 \ge 1$ and $m_n \ge 1$. Prove:

$$f[x_0^{\langle m_0 \rangle}, \ldots, x_n^{\langle m_n \rangle}] = \frac{f[x_0^{\langle m_0 \rangle}, \ldots, x_n^{\langle m_n-1 \rangle}] - f[x_0^{\langle m_0-1 \rangle}, \ldots, x_n^{\langle m_n \rangle}]}{x_0 - x_n}$$

   Prove this by induction on m_0, using parts 1, 2, and 3.

5. Prove:

$$\lim_{x_0 \to x_n} f[x_0^{\langle m_0 \rangle}, x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}] = f[x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_0+m_n \rangle}]$$

   Show first for the case $m_0 = m_n = 1$. Then show for the case $m_0 = 1$, $m_n \ge 1$ by showing

$$\lim_{x_0 \to x_n} f[x_0, x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}] = \frac{1}{m_n} \frac{\partial}{\partial x_n} f[x_1^{\langle m_1 \rangle}, \ldots, x_n^{\langle m_n \rangle}]$$

   and using part 1. Finally show for the case $m_0 \ge 1$, $m_n \ge 1$.