Exercise 6.1. This is a big problem which establishes the basic theory for the case of Lagrange interpolation with knots of arbitrary multiplicity.

1. Let $x_i$ be a knot of multiplicity $m_i$, $0 \leq i \leq n$, with the $x_i$ distinct. Then

$$f[x_0^{(m_0)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{1}{\prod_{k=0}^{n} (m_k - 1)!} \frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}} \cdots \frac{\partial^{m_n-1}}{\partial x_n^{m_n-1}} f[x_0, x_1, \ldots, x_n]$$

Prove:

$$f[x_0^{(m_0)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{1}{m_0 - 1} \frac{\partial}{\partial x_0} f[x_0^{(m_0-1)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}]$$

2. Let $m_0 = m_n = 1$. Prove:

$$f[x_0, x_1^{(m_1)}, \ldots, x_{n-1}^{(m_{n-1})}, x_n] = \frac{f[x_0, x_1^{(m_1)}, \ldots, x_{n-1}^{(m_{n-1})}] - f[x_1^{(m_1)}, \ldots, x_{n-1}^{(m_{n-1})}, x_n]}{x_0 - x_n}$$

Use direct substitution.

3. Let $m_0 = 1$ and $m_n \geq 1$. Prove:

$$f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{f[x_0, x_1^{(m_1)}, \ldots, x_{n}^{(m_{n-1})}] - f[x_1^{(m_1)}, \ldots, x_{n}^{(m_{n})}]}{x_0 - x_n}$$

Prove this by induction on $m_n$, using parts 1 and 2.

4. Let $m_0 \geq 1$ and $m_n \geq 1$. Prove:

$$f[x_0^{(m_0)}, \ldots, x_n^{(m_n)}] = \frac{f[x_0^{(m_0)}, \ldots, x_n^{(m_{n-1})}] - f[x_0^{(m_0-1)}, \ldots, x_n^{(m_n)}]}{x_0 - x_n}$$

Prove this by induction on $m_0$, using parts 1, 2, and 3.

5. Prove:

$$\lim_{x_0 \to x_n} f[x_0^{(m_0)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = f[x_1^{(m_1)}, \ldots, x_n^{(m_{n}+m_n)}]$$

Show first for the case $m_0 = m_n = 1$. Then show for the case $m_0 = 1$, $m_n \geq 1$ by showing

$$\lim_{x_0 \to x_n} f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{1}{m_n} \frac{\partial}{\partial x_n} f[x_1^{(m_1)}, \ldots, x_n^{(m_n)}]$$

and using part 1. Finally show for the case $m_0 \geq 1$, $m_n \geq 1$. 