Exercise 6.1. This is a big problem which establishes the basic theory for the case of Lagrange interpolation with knots of arbitrary multiplicity.

1. Let \( x_i \) be a knot of multiplicity \( m_i, 0 \leq i \leq n \), with the \( x_i \) distinct. Then

\[
f[x_0^{(m_0)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{1}{\prod_{k=0}^n (m_k - 1)!} \frac{\partial^{m_0-1}}{\partial x_0^{m_0-1}} \cdots \frac{\partial^{m_n-1}}{\partial x_n^{m_n-1}} f[x_0, x_1, \ldots, x_n]
\]

Prove:

\[
f[x_0^{(m_0)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{1}{m_0 - 1} \frac{\partial}{\partial x_0} f[x_0^{(m_0-1)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}]
\]

2. Let \( m_0 = m_n = 1 \). Prove:

\[
f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n-1)}] = \frac{f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n-1)}] - f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n-1)}, x_n]}{x_0 - x_n}
\]

Use direct substitution.

3. Let \( m_0 = 1 \) and \( m_n \geq 1 \). Prove:

\[
f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n-1)}] - f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n-1)}, x_n]}{x_0 - x_n}
\]

Prove this by induction on \( m_n \), using parts 1 and 2.

4. Let \( m_0 \geq 1 \) and \( m_n \geq 1 \). Prove:

\[
f[x_0^{(m_0)}, \ldots, x_n^{(m_n)}] = \frac{f[x_0^{(m_0)}, \ldots, x_n^{(m_n-1)}] - f[x_0^{(m_0-1)}, \ldots, x_n^{(m_n)}]}{x_0 - x_n}
\]

Prove this by induction on \( m_0 \), using parts 1, 2, and 3.

5. Prove:

\[
\lim_{x_0 \to x_n} f[x_0^{(m_0)}, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = f[x_1^{(m_1)}, \ldots, x_n^{(m_n)}]
\]

Show first for the case \( m_0 = m_n = 1 \). Then show for the case \( m_0 = 1, m_n \geq 1 \) by showing

\[
\lim_{x_0 \to x_n} f[x_0, x_1^{(m_1)}, \ldots, x_n^{(m_n)}] = \frac{1}{m_n} \frac{\partial}{\partial x_n} f[x_1^{(m_1)}, \ldots, x_n^{(m_n)}]
\]

and using part 1. Finally show for the case \( m_0 \geq 1, m_n \geq 1 \).