Exercise 5.1. In this problem, we will compute several linear approximations to the function \( f(x) = \sqrt{x} \) on the interval \( 0 \leq x \leq 1 \). Let \( S \) be the 2-dimensional space of linear polynomials. Use the basis functions \( \phi_1(x) = 1 \) and \( \phi_2(x) = x \).

1. Compute the linear interpolant which interpolates \( f(x) \) at \( x = 0 \) and \( x = 1 \).
2. Define the \( L^2 \) norm by
   \[
   (f, g) = \int_0^1 fg \, dx, \quad ||f||_2 = \sqrt{(f, f)}.
   \]
   Compute the best approximation \( f^* \in S \)
   \[
   ||f - f^*||_2 = \min_{v \in S} ||f - v||_2
   \]
3. Define the \( L^\infty \) norm by
   \[
   ||f||_\infty = \max_{0 \leq x \leq 1} |f(x)|
   \]
   Compute the best approximation \( f^\infty \in S \)
   \[
   ||f - f^\infty||_\infty = \min_{v \in S} ||f - v||_\infty
   \]

Exercise 5.2. In this problem we will consider the case of Hermite interpolation. In this problem, we are given \( n + 1 \) distinct knots \( x_i \), \( 0 \leq i \leq n \) with corresponding function values \( f(x_i) \) and derivative values \( f'(x_i) \). Let \( S \) be the space of polynomials of degree \( 2n + 1 \) (note the dimension of this space is \( 2n + 2 \)). We will analyze the interpolant \( f^* \in S \) which satisfies the interpolation conditions \( f^*(x_i) = f(x_i) \) for \( 0 \leq i \leq n \), and \( f'^*(x_i) = f'(x_i) \) for \( 0 \leq i \leq n \).

1. Find a value function \( V_i(x) \in S \) that satisfies \( V_i(x_j) = \delta_{ij} \) and \( V_i'(x_j) = 0 \). (Try a function of the form \( V_i(x) = L_i^2(x)(ax + b) \), where \( L_i(x) \) is the Lagrange nodal basis function, and a polynomial of degree \( n \), defined in class and \( a \) and \( b \) are coefficients to be determined)
2. Find a slope function \( S_i(x) \in S \) that satisfies \( S_i(x_j) = 0 \) and \( S_i'(x_j) = \delta_{ij} \). (Again, try a function of the form \( S_i(x) = L_i^2(x)(cx + d) \)).
3. Show

\[ f^* = \sum_{i=0}^{n} f(x_i)V_i(x) + f'(x_i)S_i(x) \]

4. Prove for \( f \in C^{2n+2} \)

\[ f(x) - f^*(x) = \frac{f^{2n+2}(\xi_x)}{(2n+2)!} \omega(x) \quad \text{where} \quad \omega(x) = \prod_{i=0}^{n}(x - x_i)^2 \]

**Exercise 5.3.** Let \( f(x) = x^3 \), and consider the knots \( x_0 = -1, x_1 = 0, x_2 = 1 \).

1. Compute the quadratic interpolant using the nodal Lagrange basis functions.
2. Compute a divided difference table for the data. Evaluate the interpolant using the upper diagonal, the lower diagonal, and a third path of your choice through the center portion of the difference table. Verify that all interpolants are the same function.
3. Add the knot \( x_3 = -2 \) to the difference table. Update the interpolant. Then add the knot \( x_4 = 2 \) to the difference table, and update the interpolant once more. Explain your results.

**Exercise 5.4.** Let \( I \) denote the interpolation *operator*, taking as argument a function \( f \) and yielding the Lagrange interpolant \( I(f) = f^* \). Prove the \( I \) is a *linear* operator. (i.e., that \( I(\alpha f + \beta g) = \alpha I(f) + \beta I(g) \), where \( \alpha \) and \( \beta \) are scalars and \( f \) and \( g \) are functions).