Math 270b: Numerical Approximation and Nonlinear Equations

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Winter Quarter 2020

Homework Assignment #5
Due Friday, February 7, 2020

Exercise 5.1. In this problem, we will compute several linear approximations to the function $f(x) = \sqrt{x}$ on the interval $0 \leq x \leq 1$. Let $\mathcal{S}$ be the 2-dimensional space of linear polynomials. Use the basis functions $\phi_1(x) = 1$ and $\phi_2(x) = x$.

1. Compute the linear interpolant which interpolates $f(x)$ at $x = 0$ and $x = 1$.
2. Define the $\mathcal{L}^2$ norm by

$$(f, g) = \int_0^1 f g \, dx, \qquad \|f\|_2 = \sqrt{(f, f)}.$$

   Compute the best approximation $f^* \in \mathcal{S}$

$$\|f - f^*\|_2 = \min_{v \in \mathcal{S}} \|f - v\|_2$$

3. Define the $\mathcal{L}^\infty$ norm by

$$\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|$$

   Compute the best approximation $f^\infty \in \mathcal{S}$

$$\|f - f^\infty\|_\infty = \min_{v \in \mathcal{S}} \|f - v\|_\infty$$

Exercise 5.2. In this problem we will consider the case of *Hermite* interpolation. In this problem, we are given $n+1$ distinct knots x_i, $0 \leq i \leq n$ with corresponding function values $f(x_i)$ *and* derivative values $f'(x_i)$. Let $\mathcal{S}$ be the space of polynomials of degree $2n+1$ (note the dimension of this space is $2n+2$). We will analyze the interpolant $f^* \in \mathcal{S}$ which satisfies the interpolation conditions $f^*(x_i) = f(x_i)$ for $0 \leq i \leq n$, and $f^{*'}(x_i) = f'(x_i)$ for $0 \leq i \leq n$.

1. Find a *value* function $V_i(x) \in \mathcal{S}$ that satisfies $V_i(x_j) = \delta_{ij}$ and $V_i'(x_j) = 0$. (Try a function of the form $V_i(x) = L_i^2(x)(ax+b)$, where $L_i(x)$ is the Lagrange nodal basis function, and a polynomial of degree n, defined in class and a and b are coefficients to be determined)
2. Find a *slope* function $S_i(x) \in \mathcal{S}$ that satisfies $S_i(x_j) = 0$ and $S_i'(x_j) = \delta_{ij}$. (Again, try a function of the form $S_i(x) = L_i^2(x)(cx+d)$).

3. Show

$$f^* = \sum_{i=0}^{n} f(x_i)V_i(x) + f'(x_i)S_i(x)$$

4. Prove for $f \in \mathcal{C}^{2n+2}$

$$f(x) - f^*(x) = \frac{f^{2n+2}(\xi_x)}{(2n+2)!}\omega(x) \quad \text{where} \quad \omega(x) = \prod_{i=0}^{n}(x - x_i)^2$$

Exercise 5.3. Let $f(x) = x^3$, and consider the knots $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.

1. Compute the quadratic interpolant using the nodal Lagrange basis functions.
2. Compute a divided difference table for the data. Evaluate the interpolant using the upper diagonal, the lower diagonal, and a third path of your choice through the center portion of the difference table. Verify that all interpolants are the same function.
3. Add the knot $x_3 = -2$ to the difference table. update the interpolant. Then add the knot $x_4 = 2$ to the difference table, and update the interpolant once more. Explain your results.

Exercise 5.4. Let $\mathcal{I}$ denote the interpolation *operator*, taking as argument a function f and yielding the Lagrange interpolant $\mathcal{I}(f) = f^*$. Prove the $\mathcal{I}$ is a *linear* operator. (i.e., that $\mathcal{I}(\alpha f + \beta g) = \alpha\mathcal{I}(f) + \beta\mathcal{I}(g)$, where α and β are scalars and f and g are functions).