**Math 270B**: Numerical Approximation and Nonlinear Equations

Instructor: Randolph E. Bank

Winter Quarter 2018

Homework Assignment #4
Due Friday, February 2, 2018

**Exercise 4.1.** The *primal* form of a linear program is:

\[
\begin{align*}
\min_{Ax\geq b} & \quad c^t x \\
\end{align*}
\]

The *dual* problem is:

\[
\begin{align*}
\max_{A^t y = c, y \geq 0} & \quad b^t y \\
\end{align*}
\]

In this problem we illustrate the connection between primal and dual formulations. We assume \( A \) has full rank.

a. Find necessary conditions for the solution of the primal problem. In particular,

\[
\begin{align*}
Ax^* & \geq b, \hat{A}x^* = \hat{b} \\
c & = \hat{A}^t \lambda^* \\
\lambda^* & \geq 0
\end{align*}
\]

where \( x^* \) is the solution, \( \hat{A} \) corresponds to active constraints at the solution and \( \lambda^* \) are the corresponding Lagrange multipliers.

b. We show that \( y^* \) given by

\[
y^* = \begin{pmatrix} \lambda^* \\ 0 \end{pmatrix}
\]

is the solution of the dual problem.

Assume \( A \) be can be partitioned as \( A^t = (\hat{A}^t \bar{A}^t) \). Verify that \( y^* \) is feasible.

\[
\begin{align*}
A^t y^* & = c \\
y^* & \geq 0
\end{align*}
\]

The active constaints for the dual problem are all of the equality constraints plus the inequality constraints for which equality holds. We may express this as

\[
\begin{pmatrix} \hat{A}^t & \bar{A}^t \end{pmatrix} \begin{pmatrix} \lambda^* \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}.
\]
c. Noting that $\max b^t y = \min -b^t y$, verify that the Lagrange multipliers $\sigma$ for the dual problem satisfy
\[
\begin{pmatrix}
\hat{A} & 0 \\
\bar{A} & I
\end{pmatrix}
\begin{pmatrix}
\hat{\sigma} \\
\bar{\sigma}
\end{pmatrix} = -b = - \begin{pmatrix}
\hat{b} \\
\bar{b}
\end{pmatrix}
\]
Thus verify
\[
\hat{\sigma} = -x^* \\
\bar{\sigma} = \bar{A}x^* - \bar{b} > 0
\]

d. Summarize the necessary conditions for the solution of the dual problem, and show that they are satisfied by $y^*$ and $\sigma$ using parts b and c.