MATH 270B: Numerical Approximation and Nonlinear Equations

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Winter Quarter 2020

Homework Assignment #3
Due Friday, January 24, 2020

Exercise 3.1. In this problem we consider the approximate solution of the two boundary value problem

$$-u'' + e^{u-v} - e^{w-u} - f = 0$$

for $0 < x < 0.15$, where

$$\begin{aligned}
f(x) &= \begin{cases} f_\ell & 0 \le x \le 0.0075 \\ f_r & 0.0075 < x \le 0.15 \end{cases} \\
v(x) &= \begin{cases} \nu & 0 \le x \le 0.015 \\ 0 & 0.015 < x \le 0.15 \end{cases} \\
w(x) &= \begin{cases} \nu & 0 \le x \le 0.0025 \\ 0 & 0.0025 < x \le 0.15 \end{cases} \\
u(0) &= \nu + \ln|f_\ell| \\
u(0.15) &= -\ln|f_r|
\end{aligned}$$

Let $n = 59$, $h = 0.0025$, and $x_i = ih$, $0 \le i \le 60$. Using the *finite difference method*, we will compute a vector U, with $U_i \approx u(x_i)$ $1 \le i \le 59$. The unknowns U_i satisfy the nonlinear system of difference equations

$$G_i(U) = \frac{2U_i - U_{i+1} - U_{i-1}}{h^2} + e^{U_i - V_i} - e^{W_i - U_i} - F_i = 0$$

for $1 \le i \le 59$. Note

$$\begin{aligned}
F_i = f(x_i) &= \begin{cases} f_\ell & 1 \le i \le 3 \\ f_r & 4 \le i \le 59 \end{cases} \\
V_i = v(x_i) &= \begin{cases} \nu & 1 \le i \le 6 \\ 0 & 7 \le i \le 59 \end{cases} \\
W_i = w(x_i) &= \begin{cases} \nu & i = 1 \\ 0 & 2 \le i \le 59 \end{cases} \\
U_0 = u(0) &= \nu + \ln|f_\ell| \\
U_{60} = u(0.15) &= -\ln|f_r|
\end{aligned}$$

Solve this system of equations by Newton's method (with line search or with $\alpha_k = (1 + \kappa\|g_k\|)^{-1}$ as described in class) for the following four cases:

case	ν	f_ℓ	f_r
1	0	10^2	-10^1
2	0	10^6	-10^3
3	40	10^6	-10^3
4	100	10^6	-10^3

Note that the Jacobian matrix is a symmetric positive definite tridiagonal matrix

$$T = \begin{pmatrix} d_1 & c & & & \\ c & d_2 & c & & \\ & \ddots & \ddots & \ddots & \\ & & c & d_{58} & c \\ & & & c & d_{59} \end{pmatrix}$$

with $c = -1/h^2$ and $d_i = 2/h^2 + e^{U_i - v(x_i)} + e^{w(x_i) - U_i}$. Iterate for $maxit = 200$ iterations or until $\|U^k - U^{k-1}\|/\|U^k\| \le \epsilon$ with $\epsilon = 10^{-8}$, where U^k is the approximate solution at the k-th Newton step. Use initial guess $(v(x_i) + w(x_i))/2$. To have some idea of how the iteration is behaving, print $\|U^k - U^{k-1}\|/\|U^k\|$ for each value of k.