MATH 270B: Numerical Approximation and Nonlinear Equations

Instructor: Randolph E. Bank

Winter Quarter 2020

Homework Assignment #2
Due Friday, January 17, 2020

Exercise 2.1. Show the following (this is mainly an exercise in notation):

a. Let $f : \Re^n \to \Re$ be continuously differentiable in an open convex set $D \subset \Re^n$. Show the *directional derivative* of f at x in the direction p, defined by

$$\frac{\partial f}{\partial p}(x) \equiv \lim_{\epsilon \to 0} \frac{f(x + \epsilon p) - f(x)}{\epsilon},$$

exists and equals $\nabla f(x)^t p$. Next, show for and $x, x + p \in D$,

$$f(x + p) = f(x) + \int_0^1 \nabla f(x + sp)^t p \, ds,$$

and there exits $z \in (x, x + p)$ such that

$$f(x + p) = f(x) + \nabla f(z)^t p.$$

b. Let $f : \Re^n \to \Re$ be twice continuously differentiable in an open convex set $D \subset \Re^n$. Show for any $x \in D$ and any nonzero perturbation $p \in \Re^n$, the second directional derivative of f at x in the direction p, defined by

$$\frac{\partial^2 f}{\partial p^2}(x) \equiv \lim_{\epsilon \to 0} \frac{\frac{\partial f}{\partial p}(x + \epsilon p) - \frac{\partial f}{\partial p}(x)}{\epsilon},$$

exists and equals $p^t \nabla^2 f(x) p$. Then show for and $x, x + p \in D$, there exits $z \in (x, x + p)$ such that

$$f(x + p) = f(x) + \nabla f(x)^t p + p^t \nabla^2 f(z) p/2.$$

Exercise 2.2. Let $\phi(x) : \Re^n \to \Re$, and consider the unconstrained optimization problem

$$\min_{x \in \Re^n} \phi(x).$$

Suppose ϕ is twice continuously differentiable, and that the Hessian matrix H is uniformly symmetric and positive definite.

a. Show that the Newton direction is always a descent direction.

b. Show that the direction $p = -\nabla \phi$ is always a descent direction; this is called the *direction of steepest descent.*

Exercise 2.3. Let $f(x) : \Re^n \rightarrow \Re^n$, and consider the system of nonlinear equations

$$f(x) = 0.$$

Suppose all components of f are continuously differentiable, and that the Jacobian matrix J is uniformly symmetric and positive definite. Show that the Newton direction is always a descent direction for $f(x)^t f(x) = \|f\|^2$.