Exercise 1.1. A function \( f(x) = (x - x^*)^m h(x) \), \( m \geq 1 \), with \( h(x^*) \neq 0 \) is said to have a root of multiplicity \( m \) at \( x^* \).

a. Prove that \( x^* \) is a root of \( f(x) \) with multiplicity \( m \) if and only if \( f(x^*) = f'(x^*) = \cdots = f^{m-1}(x^*) = 0 \) and \( f^m(x^*) \neq 0 \).

b. Prove that if \( m > 1 \), then Newton’s method is only linearly convergent.

c. Show that second order convergence is recovered for the modified Newton method

\[
x_{k+1} = x_k - \frac{m f(x_k)}{f'(x_k)}.
\]

d. Show that second order convergence is recovered for the modified Newton method

\[
x_{k+1} = x_k - \frac{f^{m-1}(x_k)}{f^m(x_k)}.
\]

e. Consider \( g(x) = f(x)/f'(x) \). Show that second order convergence is recovered for the Newton method

\[
x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}.
\]

While these modifications (as well as others) speed convergence for multiple roots, they require one to know that there is a multiple root and in some cases to also to know the multiplicity \( m \). This limits their use in many practical applications.

Exercise 1.2. Consider the function \( f(x) = (x - \alpha)^m \) for \( m \geq 1 \). Here we will study the sensitivity of multiple roots in a simple case. Let \( \epsilon \) be a small perturbation, and let \( \hat{f}(x) = f(x) + \epsilon \). Analyze the zero structure of \( \hat{f} \), and explain why it is very difficult in general to accurately compute multiple roots in a floating point number system.