MATH 270B: Numerical Approximation and Nonlinear Equations

Instructor: Randolph E. Bank

Winter Quarter 2020
Final Examination
Wednesday, March 18, 2020

NAME ____________________

SIGNATURE ____________________

#1	25	
#2	25	
#3	25	
#4	25	
#5	25	
#6	25	
#7	25	
#8	25	
Total	200	

Question 1. Let $\phi(\vec{x})$ be a scalar function of the vector variable $\vec{x}$. Suppose $\phi(\vec{x})$ is continuous with continuous first and second partial derivatives, and suppose that the Hessian is symmetric and uniformly positive definite.

1. Formally define Newton's method with line search for solving the optimization problem $\min_{\vec{x}} \phi(\vec{x})$.
2. Let $\vec{p}_k$ be the Newton search direction. Show that $\partial\phi(\vec{x}_k + \alpha\vec{p}_k)/\partial\alpha < 0$ at $\alpha = 0$. Why is this fact significant for the line search?

Question 2. Let $\phi(x)$ be a scalar function of the n-vector variable x. Suppose $\phi(x)$ is continuous with continuous first and second partial derivatives, and suppose that the Hessian is symmetric and uniformly positive definite. Let A be an $m \times n$ matrix, $n > m$ of full rank. Consider the equality constrained optimization problem

$$\min_{Ax=b} \phi(x)$$

a. Formally define the Lagrangian for this problem.

b. State the necessary conditions for existence and uniqueness of a solution.

c. Derive Newton's Method (KKT system) for solving this problem.

Question 3. Let $f(x)$ be a vector function of a vector variable x. Assume $f(x)$ is continuous and differentiable, and that the Jacobian $J(x)$ is continuous in the ball $\mathcal{B} = \{x | \|x - x^*\| \leq \delta$ for some $\delta > 0$. More specifically, assume:

1. $f(x^*) = 0$.
2. $\|J(x)^{-1}\| \leq M$ for all $x \in \mathcal{B}$.
3. $\|J(x) - J(y)\| \leq \gamma \|x - y\|$ for all $x, y \in \mathcal{B}$.

Assume the sequence x_k is generated from a starting vector $x_0 \in \mathcal{B}$ using Newton's method without line search. Using Taylor's theorem, prove

$$\|e_{k+1}\| \leq \frac{M\gamma}{2} \|e_k\|^2$$

where $e_k = x^* - x_k$. Hint: $f(x) = f(y) + \int_0^1 J(\theta x + (1-\theta)y)(x-y)d\theta$

Question 4. We consider the approximation of $x^3 + 2$ on $0 \leq x \leq 2$ by quadratic polynomials. We will use the uniform mesh $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.

a. Compute the interpolant using the Lagrange nodal basis.

b. Compute the interpolant using a divided difference table.

Question 5. Let $f \in \mathcal{C}^2(I)$, $I = [a, b]$, and let $x_i = a + ih$, $0 \leq i \leq n$, $h = (b - a)/n$ be a uniform mesh on I. Let $\tilde{f}$ denote the continuous piecewise linear polynomial interpolant of f with respect to this mesh. Using the Peano Kernel Theorem, prove:

$$\|f - \tilde{f}\|_{\mathcal{L}_2(I)} \leq Ch^2 \|f''\|_{\mathcal{L}_2(I)}$$

Question 6. Suppose we are given an inner product (f, g), and corresponding norm $\|f\| = \sqrt{(f, f)}$ defined on a vector space $\mathcal{V}$. Let $\mathcal{S} \subset \mathcal{V}$ be a finite dimensional subspace. Let $f \in \mathcal{V}$, and let $f^* \in \mathcal{S}$ be the least squares approximation of f satisfying

$$\|f - f^*\| = \min_{v \in \mathcal{S}} \|f - v\|$$

Prove the orthogonality relation

$$(f - f^*, v) = 0$$

for all $v \in \mathcal{S}$ (This shows that f^* is the *orthogonal projection* of f onto $\mathcal{S}$). Hint: consider $\|f - (f^* + \epsilon v)\|^2$, where $v \in \mathcal{S}$ and $\epsilon \in \mathcal{R}^1$.

Question 7. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x)dx$$

We consider a Gaussian quadrature formula of the form

$$\mathcal{Q}(f) = w_1 f(x_1) + w_2 f(x_2)$$

a. Compute the weights w_i and the knots x_i for the Gaussian quadrature formula.

b. Compute the error $\mathcal{I}(f) - \mathcal{Q}(f)$. Be sure to explicitly evaluate the constant.

Question 8. The composite trapazoid rule $T_h(f)$ on a uniform mesh of size $h = (b-a)/n$ is known (via the Euler-Maclaurin Summation formula) to satisfy

$$I(f) = \int_a^b f(x)\,dx = T_h(f) + \sum_{k=1}^{r} c_{2k} h^{2k} \{f^{2k-1}(b) - f^{2k-1}(a)\} + O(h^{2r+2}).$$

for $f \in \mathcal{C}^{2r}[a,b]$,

$$T_h(f) = \frac{h}{2} f(x_0) + h \sum_{k=1}^{n-1} f(x_k) + \frac{h}{2} f(x_n),$$

and $x_k = a + kh$, $0 \le k \le n$. Derive a Richardson Extrapolation Procedure for computing higher order approximations to $I(f)$.