Math 270a: Numerical Linear Algebra

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Fall Quarter 2017

Homework Assignment #9
Due Wednesday, November 29, 2017

Exercise 9.1. In this problem, we will (approximately) analyze the SSOR iteration in the tridiagonal case. Let T and M be $N \times N$ tridiagonal matrices with $T_{ii} = a$, $T_{i+1i} = T_{i-1i} = b$, $M_{ii} = c$, and $M_{i+1i} = M_{i-1i} = d$.

a. Consider the matrix $G = I - M^{-1}T$. Show that the eigenvalues and eigenvectors of G satisfy

$$(M - T)x = \lambda M x$$

b. Next show the eigenvalues satisfy

$$\lambda = 1 - \frac{a + 2b\cos\theta}{c + 2d\cos\theta}$$

where $\theta = k\pi/(N+1)$, $1 \le k \le N$. (For this use Exercise 8.1b).

c. Let $a = 2$, $b = -1$, and $T = D - L - L^t$. The SSOR matrix is

$$M = \omega^{-1}(2-\omega)^{-1}(D - \omega L)D^{-1}(D - \omega L^t)$$

By direct computation, show M is tridiagonal with constant diagonals, except for M_{11}.

d. Approximating M by a tridiagonal matrix with constant diagonals, compute the (approximate) eigenvalues of G for the case $\omega = 1$ using part b. In particular, by finding the largest eigenvalue, show that the rate of convergence is $1 - \beta h^2$ where $h = 1/(N+1)$ and $\beta = O(1)$.

e. The best value of ω is known to be of the form $2 - \alpha h$. For $\omega = 2 - \alpha h$, show the rate of convergence is $1 - \hat{\beta} h$. As above, we will approximate M by a tridiagonal matrix with constant diagonals. The algebra can then be simplified by using the identity $1 - \cos\theta = 2\sin^2(\theta/2)$ and first showing that

$$\lambda = 1 - \frac{\omega(2-\omega)8\sin^2(\theta/2)}{(2-\omega)^2 + \omega 8 \sin^2(\theta/2)}$$

(you must derive this expression for yourself). Then substitute $\omega = 2 - \alpha h$, and find the largest eigenvalue with respect to θ.