Math 270a: Numerical Linear Algebra

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Fall Quarter 2017

Homework Assignment #8
Due Wednesday, November 22, 2017

Exercise 8.1. Let T by an $N \times N$ constant coefficient tridiagonal matrix with $T_{ii} = a$ and $T_{i+1i} = T_{i-1i} = b$.

a. Show by direct substitution that the eigenvalues of T are given by

$$\lambda_k = a + 2 * b \cos\left(\frac{k\pi}{N+1}\right)$$

and corresponding (normalized) eigenvectors ψ_k given by

$$\psi_k = \sqrt{\frac{2}{N+1}} \begin{pmatrix} \sin\left(\frac{k\pi}{N+1}\right) \\ \sin\left(\frac{2k\pi}{N+1}\right) \\ \sin\left(\frac{3k\pi}{N+1}\right) \\ \vdots \\ \sin\left(\frac{Nk\pi}{N+1}\right) \end{pmatrix}$$

b. Using part a, verify the decomposition

$$T = Q\Lambda Q$$

where Q is a symmetric, orthogonal matrix with columns ψ_k, and Λ is a diagonal matrix with $\Lambda_{kk} = \lambda_k$. The orthogonal matrix Q is sometimes called the *discrete sine transform*.

Exercise 8.2. Let T be the $N \times N$ tridiagonal matrix with $T_{ii} = 2$ and $T_{ii-1} = T_{ii+1} = -1$, and consider the solution of $Tx = b$. Let $T = D - L - L^t$, where D is diagonal and L is strictly lower triangular.

a. Compute the spectral radius of the Jacobi iteration

$$Dx_k = (L + L^t)x_{k-1} + b$$

b. Now consider the Gauss-Seidel method

$$(D - L)x_k = L^t x_{k-1} + b$$

Compute the spectral radius of the Gauss-Seidel iteration matrix. To do this, first reduce the problem to showing that the spectral radius is the largest value of λ such that

$$Det(\lambda D - \lambda L - L^t) = Det(\hat{T}) = 0$$

Next, symmetrize $\hat{T}$ using a diagonal similarity transformation $\bar{T} = S\hat{T}S^{-1}$. Finally, diagonalize $\bar{T}$ using the matrix Q above, $\Sigma = Q^t\bar{T}Q$.

Exercise 8.3. Compute work estimates for solving $Tx = b$ by the Jacobi and Gauss-Seidel methods, starting from an initial guess $x_0 = 0$ and reducing the initial error by 10^{-6}. Here T is a the symmetric $N \times N$ tridiagonal matrix with $T_{ii} = 2$, and $T_{ii-1} = T_{ii+1} = -1$.