Exercise 8.1. Let $T$ be an $N \times N$ constant coefficient tridiagonal matrix with $T_{ii} = a$ and $T_{i+1,i} = T_{i-1,i} = b$.

a. Show by direct substitution that the eigenvalues of $T$ are given by

$$\lambda_k = a + 2 * b \cos \left( \frac{k\pi}{N+1} \right)$$

and corresponding (normalized) eigenvectors $\psi_k$ given by

$$\psi_k = \sqrt{\frac{2}{N+1}} \begin{pmatrix} \sin \left( \frac{k\pi}{N+1} \right) \\ \sin \left( \frac{2k\pi}{N+1} \right) \\ \sin \left( \frac{3k\pi}{N+1} \right) \\ \vdots \\ \sin \left( \frac{Nk\pi}{N+1} \right) \end{pmatrix}$$

b. Using part a, verify the decomposition

$$T = QAQ$$

where $Q$ is a symmetric, orthogonal matrix with columns $\psi_k$, and $A$ is a diagonal matrix with $A_{kk} = \lambda_k$. The orthogonal matrix $Q$ is sometimes called the discrete sine transform.

Exercise 8.2. Let $T$ be the $N \times N$ tridiagonal matrix with $T_{ii} = 2$ and $T_{ii-1} = T_{ii+1} = -1$, and consider the solution of $Tx = b$. Let $T = D - L - L'$, where $D$ is diagonal and $L$ is strictly lower triangular.

a. Compute the spectral radius of the Jacobi iteration

$$Dx_k = (L + L')x_{k-1} + b$$

b. Now consider the Gauss-Seidel method

$$(D - L)x_k = L'x_{k-1} + b$$
Compute the spectral radius of the Gauss-Seidel iteration matrix. To do this, first reduce the problem to showing that the spectral radius is the largest value of $\lambda$ such that

$$Det(\lambda D - \lambda L - L^1) = Det(\hat{T}) = 0$$

Next, symmetrize $\hat{T}$ using a diagonal similarity transformation $\hat{T} = S\hat{T}S^{-1}$. Finally, diagonalize $\hat{T}$ using the matrix $Q$ above, $\Sigma = Q^TQ$.

**Exercise 8.3.** Compute work estimates for solving $Tx = b$ by the Jacobi and Gauss-Seidel methods, starting from an initial guess $x_0 = 0$ and reducing the initial error by $10^{-6}$. Here $T$ is a the symmetric $N \times N$ tridiagonal matrix with $T_{ii} = 2$, and $T_{i,i-1} = T_{i,i+1} = -1$. 