Exercise 6.1. Let $A$ be a symmetric $n \times n$ matrix, and let $q$ be a vector satisfying $\|q\|_2 = 1$ which approximates an eigenvector $v$ of $A$. Assume $\|v\|_2 = 1$. Let $v_1, v_2, \ldots, v_n$ by a set of orthonormal eigenvectors of $A$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. For convenience, suppose the $v_i$ are ordered such that $v_1 = v$. The approximate eigenvector $q$ can be expressed in terms of the $v_i$ as

$$q = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

a. Show that

$$\sum_{k=1}^{n} c_k^2 = |q|_2^2 = 1$$

b. Show that

$$\sum_{k=2}^{n} c_k^2 \leq \|v - q\|_2^2$$

c. Derive an expression for the Rayleigh quotient $\rho = q^t A q$ in terms of the coefficients $c_i$ and the eigenvalues $\lambda_i$.

d. Show that

$$|\lambda_1 - \rho| \leq C \|v - q\|_2^2$$

where

$$C = \max_{2 \leq k \leq n} |\lambda_k - \lambda_1|$$

(Hint: $\lambda_1 = \sum_{k=1}^{n} \lambda_1 c_k^2$).

Thus, if $\|v - q\|_2 = O(\epsilon)$, then $|\lambda_1 - \rho| = O(\epsilon^2)$.

Exercise 6.2. Let

$$A_0 = A = \begin{pmatrix} 8 & 1 \\ -2 & 1 \end{pmatrix}$$

a. Find an orthogonal $Q_1$ (a plane rotation) and an upper triangular $R_1$ such that $A_0 = Q_1 R_1$. Calculate $A_1 = R_1 Q_1$. Notice that $A_1$ is closer to upper triangular form than $A_0$, and the main diagonal of $A_1$ approximates the eigenvalues of $A$ better than those of $A_0$ do.
b. Now perform one step of the QR algorithm with Rayleigh quotient shift. Compare your result with part a. Which is better?

Exercise 6.3. Do this exercise using Matlab. We will study the power method and its variants. Let $T$ be the $n \times n$ symmetric, positive definite tridiagonal matrix

$$T = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

a. The eigenvalues of $T$ are given by

$$\lambda_k = 2 - 2 \cos \left( \frac{k \pi}{n+1} \right)$$

$1 \leq k \leq n$. Each eigenvalue is simple and the corresponding eigenvectors are

$$v_k = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin \left( \frac{k \pi}{n+1} \right) \\ \sin \left( \frac{2k \pi}{n+1} \right) \\ \sin \left( \frac{3k \pi}{n+1} \right) \\ \vdots \\ \sin \left( \frac{nk \pi}{n+1} \right) \end{pmatrix}$$

By forming the product $Tv_k$, verify directly that $Tv_k = \lambda_k v_k$.

b. Now let $n = 31$. Apply 10 iterations of the power method using the starting vector $q_0 = [1, 1, \cdots, 1]$. Rescale so that the first component of $q_i$ is 1, $(q_i^t = [1, *, *, \cdots, *]$ where “*” is just a generic entry). Since we know the eigenvalues of $T$, compute the ratio $\lambda_{30}/\lambda_{31}$, which will give you a good idea of the rate of convergence. Compute $\|q_i - \alpha v_{31}\|^2$, $0 \leq i \leq 10$, where $\alpha = 1/(v_{31})_1$.

c. Now we will try the same problem using shifted inverse iteration. Let $\rho = \lambda_{31} + 10^{-6}$. Apply 3 iterations of the inverse power method using $T - \rho I$, again starting from $q_0 = [1, 1, \cdots, 1]$. Rescale as in part b. Compute the ratio the ratio $(\rho - \lambda_{31})/(\rho - \lambda_{30})$ to estimate the rate of convergence. Compute $\|q_j - \alpha v_{31}\|^2$, $0 \leq j \leq 3$, where $\alpha = 1/(v_{31})_1$.

d. The inverse power method is able to compute intermediate eigenvectors with equal ease. To see this, let $\rho = \lambda_{16} + 10^{-6}$. Apply 3 iterations of the inverse power method using $T - \rho I$, starting from $q_0 = [1, 1, \cdots, 1]$. Normalize as before. Compute $\alpha v_{16}$ where $\alpha = 1/(v_{16})_1$ for purposes of comparison. Compute the ratio the ratio $|(\rho - \lambda_{16})/(\rho - \lambda_{17})|$. Compute $\|q_j - \alpha v_{16}\|^2$, $0 \leq j \leq 3$, where $\alpha = 1/(v_{16})_1$.

e. The last experiment should illustrate the performance of the Rayleigh quotient iteration for different starting vectors. Do 4 steps of the Rayleigh quotient iteration starting form the vector $q_0 = [1, 1, \cdots, 1]$. Then repeat the experiment, except choose the starting vector $q_0 = [1, -1, 1, -1, \cdots, -1, 1]$. 