# Math 270a: Numerical Linear Algebra

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Fall Quarter 2017

Homework Assignment #4
Due Wednesday, October 25, 2017

**Exercise 4.1.** Let $x, y \in \Re^n$, with $\|x\|_2 = \|y\|_2 > 0$. Find a Householder transformation $Q = I - 2qq^t$ such that $Qx = y$.

**Exercise 4.2.** Let

$$A = \begin{pmatrix} 1 & 1/2 & 2 \\ \sqrt{2} & -\sqrt{2}/2 & \sqrt{2} \\ 1 & -3/2 & 4 \end{pmatrix}$$

Compute the factorization $A = QR$ where $Q$ is orthogonal and $R$ is upper triangular in two ways:

a. Use three plane rotations.

b. Use two Householder transformations.

In both cases, compute the transformations such that the diagonal entries of $R$ will be positive, and form the matrix $Q$ as the product of the elementary transformations.

**Exercise 4.3.** These questions refer to an orthogonal *projector* from $\Re^n$ to a subspace $\mathcal{S} \subset \Re^n$.

1. Let $P$ be a projector. Prove that for any $x \in \Re^n$

$$\|x\|_2^2 = \|Px\|_2^2 + \|(I-P)x\|_2^2.$$

2. Let $P$ be a projector. Show $P = UU^t$, where the columns of $U$ are orthogonal.
3. Let $P$ be a projector. Show $P^2 = P$ ($P$ is *idempotent*).
4. Prove if $A$ has linearly independent columns, then $A(A^tA)^{-1}A^t$ is the projector onto $Range(A)$.

**Exercise 4.4.** The classical and modified Gram-Schmidt algorithms are identical as far as the first two orthogonal vectors $q_1$ and $q_2$ are concerned, so any example which illustrates their differences must have at least 3 vectors. Let $v_1 = (1, \delta, 0, 0)^t$, $v_2 = (1, 0, \delta, 0)^t$, and $v_3 = (1, 0, 0, \delta)^t$, where $|\delta| \ll 1$. Note that these vectors are nearly linearly dependent. Suppose that $\delta$ is so small that $\delta^2 < \epsilon$, where $\epsilon$ is the unit roundoff (machine epsilon) on whatever computer is being used. Then $f\ell(1 + \delta^2) = 1$.

**a.** Compute the vectors $q_1$, $q_2$ and $q_3$ by the classical Gram-Schmidt process, making the approximation that $1 + \delta^2 \approx 1$. Next compute the inner products $(q_1, q_2)$, $(q_1, q_3)$, and $(q_2, q_3)$. Note that the computed $q_2$ and $q_3$ satisfy $(q_2, q_3) = 1/2$; thus they are far from orthogonal.

**b.** Now compute the vectors $q_1$, $q_2$ and $q_3$ by the modified Gram-Schmidt process, again making the approximation that $1 + \delta^2 \approx 1$, and then compute the inner products $(q_1, q_2)$, $(q_1, q_3)$, and $(q_2, q_3)$. Note that the modified Gram-Schmidt process has done a reasonable job of producing a set of orthonormal vectors.