Math 270a: Numerical Linear Algebra

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Fall Quarter 2017

Homework Assignment #3
Due Wednesday, October 18, 2017

Exercise 3.1. Let A be a $n \times n$ symmetric, positive definite *band* matrix with semi-bandwidth s. Let $A = LL^t$ be the Cholesky factorization of A.

a. Following the procedure used in class for the dense case, show that the cost (in flops) of computing the Cholesky factor is $ns^2/2 - s^3/3$ (for the two highest order terms). As a hint, note that the first $n - s$ steps are similar to one another, and each involves the same amount of work. The last s steps require the Cholesky factorization of a dense $s \times s$ matrix.

b. Show that cost of solving $Ax = b$, given the Cholesky factor L and exploiting the band structure, is $2ns - s^2$ (again, just the two highest order terms).

c. Suppose $n = 10,000$ and $s = 100$. Compare the cost (in flops) of band Cholesky to that of standard Cholesky (i.e., not exploiting the zeros in A). Do this for both the factorization and the solution phases.

Exercise 3.2. Use Matlab to do this problem. In this problem we will study the iterative refinement algorithm. Since Matlab doesn't allow one to directly specify the precision of a calculation, we will simulate that portion of the computation. Let T be the 10×10 symmetric, positive definite tridiagonal matrix

$$T = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

a. Compute the right hand side $b = Tx$, where $x = [1\,1\,1\,\ldots 1]^t$.

b. Compute the Cholesky factorization $T = LL^t$.

c. Compute an approximate Cholesky factor $\hat{L} = \mathrm{ceil}(100.0 * L)/100.0$. *Ceil* is a Matlab function which rounds to an integer. This has the effect of truncating the factorization to roughly two significant digits, and simulates the effect of having used low precision arithmetic to compute it. For the remainder of the exercise, we will use the approximate Cholesky factor $\hat{L}$.

d. Compute $\delta T = \hat{L}\hat{L}^t - T$, $\|\delta T\|/\|T\|$, and $Cond(T)$.

e. Let $x_0 = 0$. Compute x_i, $1 \le i \le 4$, using iterative refinement:

$$\begin{aligned} r_{i-1} &= b - Tx_{i-1} \\ \hat{L}y_{i-1} &= r_{i-1} \\ \hat{L}^t e_{i-1} &= y_{i-1} \\ x_i &= x_{i-1} + e_{i-1} \end{aligned}$$

f. Compute the relative error $\|x - x_i\|/\|x\|$, $1 \le i \le 4$. Explain the observed behavior of the error.

Exercise 3.3. Let A be an $n \times m$ with $m \le n$ and rank m. Let L be the lower triangular matrix which arising from the Cholesky factorization of A^tA (i.e., $A^tA = LL^t$). Let R be the $m \times m$ upper triangular matrix arising from the QR factorization of A (i.e. $A = QR$ where Q is $n \times m$ with orthogonal columns). Prove $L^t = DR$ where D is a diagonal matrix with diagonal entries ± 1.