Exercise 2.1. Let $A$ be an $n \times n$ nonsingular matrix. Prove $PA = LDU$, where $P$ is a permutation matrix, $L$ is unit lower triangular, $D$ is diagonal and $U$ is unit upper triangular.

Exercise 2.2. Let $A$ be an $n \times n$ symmetric, positive definite matrix and consider the partition

$$A = \begin{pmatrix} \alpha & c^t \\ c & B \end{pmatrix},$$

where $\alpha$ is a scalar and $B$ is $n - 1 \times n - 1$. Prove the $n - 1 \times n - 1$ matrix

$$S = B - \frac{cc^t}{\alpha}$$

is also symmetric positive definite.

Exercise 2.3. Let $A$ be an $n \times n$ symmetric positive definite matrix. Prove $A = LDL^t$, where $L$ is unit lower triangular and $D$ is diagonal with positive diagonal entries. (Hint: use Exercise 2.2 and induction.)

Exercise 2.4. Let $A$ be an $n \times n$ symmetric positive definite matrix. Prove $A = LL^t$, where $L$ is lower triangular with positive diagonal entries. This factorization is called the Cholesky factorization.

Exercise 2.5. Let $A$ be an $n \times n$ symmetric, positive definite matrix, and consider the partition

$$A = \begin{pmatrix} B & c \\ c^t & \alpha \end{pmatrix},$$

where $B$ is $n - 1 \times n - 1$ and $\alpha$ is a scalar. Let $B$ have the decomposition $B = \bar{L}D\bar{L}^t$. Use this decomposition to compute the $LDL^t$ decomposition of $A$. This is another variant of Gaussian elimination called bordering.