Math 270a: Numerical Linear Algebra

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Fall Quarter 2017

Homework Assignment #1
Due Wednesday October 4, 2017

Exercise 1.1. Let $\|\cdot\|$ be any vector norm and $A \in \Re^{m\times n}$. For $x \in \Re^n$, let $\|x\|_A = \|Ax\|$.

a. Prove $\|x\|_A$ is a seminorm.

b. Prove if $Rank(A) = n$, then $\|x\|_A$ is a norm.

Exercise 1.2. Let $A \in \Re^{n\times n}$ be symmetric and positive definite. For $x, y \in \Re^n$, let $(x, y)_A = x^t Ay$. Prove $(\cdot, \cdot)_A$ is an inner product.

Exercise 1.3. Let V be a vector space with inner product (x, y) for $x, y \in V$. For $x \in V$, define $\|x\| \equiv \sqrt{(x, x)}$. Prove $\|\cdot\|$ is a norm.

Exercise 1.4. Let $A \in \Re^{m\times n}$. Prove:

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |A_{ij}|$$

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |A_{ij}|$$

Exercise 1.5. Show that for a given norm, $Cond(AB) \leq Cond(A)Cond(B)$, and that $Cond(\alpha A) = Cond(A)$ for all nonzero α.

Exercise 1.6. Try this computer exercise using Matlab. The *Hilbert matrix* H of order n is defined by

$$H_{ij} = \int_0^1 x^{i-1}x^{j-1}\,dx = \frac{1}{i+j-1}$$

This matrix is the *mass matrix* or *gram matrix* for the monic polynomials x^k, $0 \leq k \leq n-1$. It is easy to see that H is symmetric and positive definite. In particular, for $v \neq 0$,

$$v^t Hv = \int_0^1 p(x)^2\,dx > 0$$

where

$$p(x) = \sum_{i=1}^{n} v_i x^{i-1}$$

Clearly $p(x) \equiv 0$ if and only if $v = 0$. For $n = 3$, $n = 6$, $n = 9$ and $n = 12$, make the following computations with the Hilbert matrix (in Matlab, try the command $H = hilb(n)$ to generate the matrix).

a. Form the vector $b = He$, where $e^t = [1, 1, \ldots, 1]$.

b. Solve the linear system $Hx = b$. Compare the vectors e and x, which should be equal in exact arithmetic.

c. Compute the condition number of H.

d. Explain the observed behavior.