Math 270C: Numerical Ordinary Differential Equations

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PRACTICE/SAMPLE FINAL EXAM

#1	10	
#2	10	
#3	10	
#4	10	
#5	10	
#6	10	
#7	10	
#8	10	
#9	10	
#10	10	
Total	100	

Name ____

Instructions:

- No calculators allowed (or needed).
- No crib sheets or notes allowed.
- You must show your work.
- Put your name on the line above and staple this sheet to the pages containing your exam solutions.

Questions:

1. (Gaussian Quadrature: 10 points.) Let $P_k(x)$, $k = 0, 1, \ldots$ be an orthgonal family of polynomials, and let $x_0, x_1, \ldots, x_k$ be the $k+1$ distinct zeros of $P_{k+1}(x)$.

   (a) Prove mutual orthogonality of the Lagrange polynomials:

   $$L_{k,i}(x) = \prod_{j=0, j\neq i}^{k} \frac{(x - x_j)}{(x_i - x_j)}, \qquad i = 0, \ldots, k.$$

   (b) Show positivity of the Gauss weights: $A_i = \int_a^b L_{k,i}(x) w(x) dx$.

   (c) Determine a quadrature rule, exact for polynomials of degree $n \leq 2$, of the form:

   $$\int_{-1}^{1} f(x)dx \approx A_0 f(-\frac{1}{2}) + A_1 f(0) + A_2 f(\frac{1}{2}).$$

2. (Best Approximation: 10 points) Consider the problem of best L^p-approximation of a (continuous) function $u(x)$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$: Find $u^* \in V$ such that

   $$\|u - u^*\|_{L^p} = \inf_{v \in V} \|u - v\|_{L^p},$$

   where

   $$\|u\|_{L^p} = \left(\int_0^1 |u|^p \, dx \right)^{1/p}, \; 1 \leq p < \infty, \qquad \|u\|_{L^\infty} = \sup_{x\in[0,1]} |u(x)|.$$

   We wish to find the best L^p-approximation of the specific function $u(x) = x^4$.

   (a) Determine the best L^2-approximation in the subspace of linear functions; i.e., $V = \text{span}\{1, x\}$, and justify the technique you use.

   (b) Why (specifically) does this problem become much more difficult if we consider the case $p \neq 2$?

   (c) Consider now replacing L^2 with H^1, which is the Hilbert space equipped with the following inner-product:

   $$(u, v)_{H^1} = \int_0^1 \left(\frac{du}{dx}\frac{dv}{dx} + uv \right) dx,$$

   which then induces the norm:

   $$\|u\|_{H^1} = (u, u)_{H^1}^{1/2}.$$

   Determine the best H^1-approximation in the subspace of linear functions; i.e., $V = \text{span}\{1, x\}$, and again justify the technique you use.

3. (IVP in ODE: 10 points.) Consider the following initial value problem (IVP):

$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha.$$

Let $f(t, y) = t^3 y - 2$, $a = 0$, $b = 1$, $\alpha = 1$.

(a) Write down the definition of a *well-posed problem.*

(b) Rigorously prove that this problem is well-posed for this choice of $f(t, y)$, a, b, and α.

4. (IVP on ODE: 10 points.) Consider the following class of one-step methods ($\theta \in [0, 1]$) for the IVP in Problem 3:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h[\theta f(t_i, w_i) + (1 - \theta) f(t_{i+1}, w_{i+1})] \end{aligned}$$

(a) Determine truncation error for this class of methods.

(b) For $\theta = 1$, is the method consistent and stable? What is the region of stability?

(c) For $\theta = 0$, is the method consistent and stable? What is the region of stability?
Hint: We really only answer this for the model problem: $y' = \lambda y$, $\lambda < 0$, $y(0) = \alpha$.

(c) What ranges of θ make the method of Problem 4 stable, unstable, and/or conditionally stable?

5. (IVP in ODE: 10 points.) Recall that one of several Runge-Kutta methods of order two for the IVP in Problem 3 above was the *Midpoint Method*:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right). \end{aligned}$$

(a) Apply this RK-2 method to the specific IVP in Problem 3 above to produce a numerical approximation to the solution. Use $h = 0.5$ in your application of the method, and just do the first step to give you an approximation at $t = 0.5$.

(b) Write down the definition of local truncation error for a general one-step method such as this one.

(c) Show explicitly that the local truncation error of the RK-2 method in Problem 5 is $O(h^2)$.
Hint: Use a 2D Taylor expansion of $a_1 f(t + \alpha_1, y + \beta_1)$ *to match the leading terms in the order-2 Taylor method, where* $T^{(2)}(t, y) = f(t, y) + \frac{h}{2}\frac{\partial f}{\partial t}(t, y) + \frac{h}{2}\frac{\partial f}{\partial y}(t, y) \cdot f(t, y)$.

6. (Systems of ODE: 10 points.) Consider the following fourth-order IVP:

$$y'''' = 4y''' - y'' - 2y' + 3y, \qquad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1.$$

(a) Transform the ODE into an equivalent 4-component first-order ODE.

(b) To produce a completely specified 4-component first-order IVP in ODE, specify the initial conditions for each component of the system.

7. (Systems of ODE: 10 points.) Prove that the IVP in Problem 6 is well-posed.

Hint: Show that the vector-valued function F that you had to construct to build the first-order system is Lipschitz with respect to the Euclidean 2-norm of a vector.

8. (BVP in ODE: 10 points.) Consider the following two-point boundary-value problem (BVP),

$$-(au_x)_x + b = f, \quad c \le x \le d, \quad u(c) = \alpha, \quad u(d) = \beta,$$

where $a, b, f\colon [c,d] \to \mathbb{R}$, with $f \in L^2(c,d)$, and

$$0 \le b(x) \le b_1 < \infty, \qquad 0 < a_0 \le a(x) \le a_1 < \infty, \qquad \forall x \in [c,d].$$

(a) Assuming the simple case $a(x) = 1$, $b(x) = 0$, $c = 0$, $d = 1$, use the finite difference mesh

$$c = x_0 < x_1 < \cdots < x_N < x_{N+1} = d, \qquad x_i = x_{i-1} + h = c + ih, \qquad h = \frac{(d-c)}{N+1},$$

to show that the centered-difference finite difference approximation $w_i \approx u(x_i)$, with $O(h^2)$ accuracy, satisfies the following matrix equation:

$$AU = F,$$

where

$$A = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}, \qquad F = \begin{pmatrix} h^2 f(x_1) - \alpha \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_N) - \beta \end{pmatrix}.$$

In other words, you must derive the matrix system above by applying the finite different method to the BVP. To receive full credit, you must show all steps, including the step which incorporates the boundary conditions into the matrix system. (You do not have to solve this linear system.)

(b) Repeat part (a), but allow for variable mesh spacing:

$$c = x_0 < x_1 < \cdots < x_N < x_{N+1} = d, \qquad x_i = x_{i-1} + h_i, \qquad h_i > 0,$$

showing how the entries of A and F now must in corporate the spacings h_i. (Again, you do not have to solve this linear system.)

9. (BVP in ODE: 10 points.) Consider the BVP in ODE from Problem 8.

(a) Showing each step, derive the following weak formulation the BVP:

$$\text{Find } u \in H_0^1(c,d) \text{ such that } A(u,v) = F(v), \quad \forall v \in H_0^1(c,d),$$

where

$$A(u,v) = \int_c^d au_x v_x + buv\ dx, \qquad F(v) = \int_c^d fv dx - A(\bar{u}, v), \qquad \bar{u}(c) = \alpha, \quad \bar{u}(d) = \beta.$$

(b) Prove that the weak formulation of the problem in part (a) is well-posed.

(c) Use the same uniform partitioning of the interval $[c,d]$ from from part (a) of Problem 8 to build a piecewise linear basis for a subspace $S \subset H_0^1(c,d)$. Now use this basis with Galerkin's method and the weak formulation from part (a) of this problem to produce a linear system for the coefficients of a Galerkin approximation in this basis. (As in Problem 8, you do not have to solve this linear system.)

10. (BVP in ODE: 10 points.) State and prove "Cea's Lemma" to give an *a priori* error estimate for your Galerkin approximation in Problem 9.

* (Best Approximation: CHALLENGING/EXTRA CREDIT)

In this problem, we will prove existence and uniqueness of best approximation, in the case of a Hilbert space H and a closed convex subset $M \subset H$. Prove the following:

Theorem. Let M be a closed convex subset of a Hilbert space H. Then for every $x \in H$, $\exists!\ y \in M$ such that

$$\|x - y\|_H = \inf_{w \in M} \|x - w\|_H \equiv \text{dist}(x, M) \equiv d.$$

Part 1 of Proof. Given an argument justifying the assumption that there exists a sequence $\{y_n\}$, $y_n \in M$, such that

$$\lim_{n \mapsto \infty} \|x - y_n\|_H = \inf_{w \in M} \|x - w\|_H.$$

Hint for Part 1: Use the definition of the *inf.*

Part 2 of Proof. Argue that:

$$\|x - \frac{1}{2}(y_m + y_n)\|_H \geq d = \inf_{w \in M} \|x - w\|_H, \qquad \forall m, n \in \mathbb{N}.$$

Hint for Part 2: Use convexity of M.

Part 3 of Proof. Show that:

$$\|y_m - y_n\|_H^2 = 2\|x - y_m\|_H^2 + 2\|x - y_n\|_H^2 - 4\|x - \frac{1}{2}(y_m + y_n)\|_H^2.$$

Hint for Part 3: Use the Parallelogram Law (with the right choice of arguments). This is where having a Hilbert space leads to an easier proof than having only a Banach space.

Part 4 of Proof. Prove that the minimizing sequence $\{y_n\}$ is in fact Cauchy, that it converges to a $y \in M$, and that:

$$\|x - y\|_H = d.$$

Note that this $y \in M$ is the best approximation that we were looking for, but we don't know if it is unique.

Hint for Part 4: Use Parts 1, 2, 3, and a property of the norm to show the Cauchy property for $\{y_n\}$. Then use what you know about H and M to establish the existence of the limit point $y \in M$, and finally, use the continuity of the norm (proof given in class) to establish the inequality above.

Part 5 of Proof. Prove that this best approximation $y \in M$ is in fact unique.

Hint for Part 5: Use the standard technique of assuming there are two best approximations y and $\bar{y}$, and then show they must be the same. The key trick in showing this is to consider $\frac{1}{2}(y + \bar{y}) \in M$, and then use the Parallelogram Law again, along with some properties of the norm.

* (Piecewise Polynomial Approximation: CHALLENGING/EXTRA CREDIT)

Let $u(x)$ be analytic on $[a, b]$, and consider a subdivision of the interval as $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ using $n + 1$ nodal points. We will not assume uniformity, so that the points may be placed as desired in the interval (but they must remain distinct). Define $h = max_i h_i$, where $h_i = x_i - x_{i-1}, i = 1, \ldots, n + 1$.

(a) Precisely define the $n + 1$ continuous piecewise linear Lagrange functions $\phi_i(x)$ as follows. There are $n + 1$ "hat" functions satisfying the Lagrange property at the $n + 1$ nodal points:

$$\phi_i(x_j) = \delta_{ij}, \qquad i = 0, \ldots, n.$$

After you have defined (given explicit formulae) the hat functions, sketch a picture of a few neighboring functions.

(b) Let u_I denote the continuous piecewise linear polynomial interpolant of $u(x)$ on $[a, b]$, defined using the basis above. Prove the following fundamental L^2-error estimate for the continous piecewise linear interpolant:

$$\|u - u_I\|_{L^2} \leq Ch^2 \|u\|_{H^2}, \text{ where } \|u\|_{H^k} = \left(\int_a^b \sum_{i=0}^{k} \left| \frac{d^i u}{dx^i} \right|^2 dx \right)^{1/2}.$$

(c) Prove the following fundamental H^1-error estimate:

$$\|u - u_I\|_{H^1} \leq Ch \|u\|_{H^2}.$$

(d) Prove the following L^∞ error estimate:

$$\|u - u_I\|_{L^\infty} \leq Ch^2 \|u\|_{H^2}.$$