HW Answer 270B

1. $$l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j} \in P_n$$

for any $f(x)$. the interpolation polynomial is $\sum_{i=0}^{n} f(x_i)\, l_i(x)$

if $f(x)\in P_n$ then $f(x)$ and $\sum_{i=0}^{n} f(x_i)\, l_i(x)$ are two n-degree polynomials coincide at $n+1$ points

$\therefore$ $f(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$ $f(x)$ is recovered

So $\{l_i(x)\}_{i=0}^{n}$ spans the space P_n $\therefore$ $\{l_i(x)\}_{i=0}^{n}$ forms a basis

2.

$$\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & & \\ x_0^{n-1} & x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix} = \det\begin{pmatrix} \cancel{1} & \cancel{1} & \cdots & \cancel{1} \\ 0 & x_1-x_0 & \cdots & x_n-x_0 \\ 0 & x_1(x_1-x_0) & \cdots & x_n(x_n-x_0) \\ \vdots & & & \\ 0 & x_1^{n-2}(x_1-x_0) & \cdots & x_n^{n-2}(x_n-x_0) \\ 0 & x_1^{n-1}(x_1-x_0) & \cdots & x_n^{n-1}(x_n-x_0) \end{pmatrix} = (x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

reduce the dimension

multiply $(-x_0)$ on the kth row and add it to the $(k+1)$th row, do it from the bottom to top

exact common factors from each column $(x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)$

by induction $$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ x_0^n & x_1^n & \cdots & x_n^n \end{pmatrix} = \sum_{i<j}(x_j-x_i)$$

So Vandermonde's matrix is nonsingular iff $\{x_i\}$ are distinct.

3. $w_{n+1}(x) = \prod_{i=0}^{n} (x-x_i)$

$w_{n+1}'(x) = \sum_{j=0}^{n} \prod_{k\neq j} (x-x_k)$ So $w_{n+1}'(x_i) = \prod_{j\neq i}(x_i - x_j)$

because any term contains $(x-x_i)$ will vanish.

So $l_i(x) = \dfrac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)} = \dfrac{w_{n+1}(x)}{(x-x_i)\, w_{n+1}'(x_i)}$

6. $w_{n+1}(x) = (x+Nh)(x+(N-1)h)\cdots(x+h)\,x\,(x-h)(x-2h)\cdots(x-Nh)$

$\overset{x=rh}{=\!=} h^{n+1}(N+r)(N-1+r)\cdots(1+r)\,r\,(r-1)(r-2)\cdots(r-N)$

$w_{n+1}(x+h) \overset{x=rh}{=\!=} h^{n+1}(N+r+1)(N+r)\cdots(2+r)(1+r)\,r\,(r-1)\cdots(r-N+1)$

So $\left|\dfrac{w_{n+1}(x+h)}{w_{n+1}(x)}\right| = \left|\dfrac{N+1+r}{r-N}\right| > 1$ for $r \in (0, N-1)$

$\therefore$ $|w_{n+1}(x)|$ attains maximum at interval (x_{n-1}, x_n)

Since each point in $(0, x_{n-1})$ is smaller than the corresponding point in (x_{n-1}, x_n).

7. $f[x_0 x_1 \cdots x_n]$ is defined as the leading coefficient of the uniquely determined n-degree polynomial at nodes $\{x_0\ x_1 \cdots x_n\}$

Consider the $(n-1)$ degree polynomial determined at $\{x_0\ x_1 \cdots x_{n-1}\}$ as $P_1(x)$ and $\{x_1 x_2 \cdots x_n\}$ as $P_2(x)$

$p(x) = P_1(x)\cdot \dfrac{x_n - x}{x_n - x_0} + P_2(x)\cdot\dfrac{x-x_0}{x_n-x_0}$ is a n-degree polynomial.

We can check that $p(x)$ is the uniquely determined n-degree polynomial at nodes $\{x_0\, x_1 \cdots x_n\}$

$$P(x_0) = P_1(x_0)\cdot\frac{x_n - x_0}{x_n - x_0} = P_1(x_0) = f(x_0) \qquad P(x_n) = P_2(x_n)\cdot\frac{x_n - x_0}{x_n - x_0} = f(x_n)$$

and for $1 \leq i \leq n-1$

$$P(x_i) = P_1(x_i)\cdot\frac{x_n - x_i}{x_n - x_0} + P_2(x_i)\frac{x_i - x_0}{x_n - x_0}$$

$$= f(x_i)\cdot\left(\frac{x_n - x_i}{x_n - x_0} + \frac{x_i - x_0}{x_n - x_0}\right) = f(x_i)$$

So $f[x_0\, x_1 \cdots x_n]$ = leading coefficient of $P(x) = f[x_0 \cdots x_{n-1}]\cdot\frac{-1}{x_n - x_0} + f[x_1 \cdots x_n]\frac{1}{x_n - x_0}$

$$= \frac{f[x_1 \cdots x_n] - f[x_0 \cdots x_{n-1}]}{x_n - x_0}$$

11. (1) $S_3(x)$ natural cubic spline interpolating f. i.e. $S_3''(a) = S_3''(b) = 0$

$$\int_a^b f''(x)^2 dx = \int_a^b [f''(x) - S''(x) + S''(x)]^2 dx$$

$$= \int_a^b S''(x)^2 dx + \int_a^b [f''(x) - S''(x)]^2 dx + 2\int_a^b S''(x)\cdot(\cancel{f''(x) - S''(x)})\,dx$$

$$\int_a^b S''(x)(f''(x) - S''(x))\,dx = \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} S''(x)(f''(x) - S''(x))\,dx$$

$$= \sum_{i=0}^{n}\left(S''(x)(f'(x) - S'(x))\Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} \underbrace{S'''(x)}_{\text{constant on each interval}}\cdot(f'(x) - S'(x))\,dx\right)$$

$$= S''(x)(f'(x) - S'(x))\Big|_a^b - \sum_{i=0}^{n} S'''(x)\cdot\int_{x_i}^{x_{i+1}} f'(x) - S'(x)\,dx$$

Since $\int_{x_i}^{x_{i+1}} f'(x) - S'(x)\,dx = (f(x) - S(x))\Big|_{x_i}^{x_{i+1}} = 0$ and $S''(a) = S''(b) = 0$

term $\int_a^b S''(x)\cdot(f''(x) - S''(x))\,dx$ vanishes $\therefore \int_a^b f''(x)^2 dx = \int_a^b S''(x)^2 dx + \int_a^b [f'' - S'']^2 dx$

$\therefore \int_a^b s''(x)^2 dx \le \int_a^b f''^2(x) dx$ "=" holds iff $f''(x) \equiv s''(x)$ on each interval. $\therefore f(x) = S(x) + (ax+b)$

Since $f(x)$ and $S(x)$ match at the endpoints. $f(x) \equiv S(x)$

minimal energy property of natural cubic spline interpolation.

(2) $S_f'(a) = f'(a)$ $S_f'(b) = f'(b)$

$$\int_a^b [f''(x) - s''(x)]^2 dx = \int_a^b [f''(x) - S_f''(x) + S_f''(x) - s''(x)]^2 dx$$

$$= \int_a^b [f''(x) - S_f''(x)]^2 dx + \int_a^b [S_f''(x) - s''(x)]^2 dx + 2\int_a^b \cancel{[f'' - S_f''](S_f'' - s'')} dx$$

$$\int_a^b (f'' - S_f'')(S_f'' - s'') dx = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (f'' - S_f'')(S_f'' - s'') dx$$

$$= \sum_{i=0}^{n} (f' - S_f')(S_f'' - s'')\Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} (f' - S_f')(S_f''' - s''') dx$$

$$= \cancel{(f' - S_f')(S_f'' - s'')}\Big|_a^b - \sum_{i=0}^{n} (S_f''' - f''') \int_{x_i}^{x_{i+1}} \cancel{(f' - S_f')} dx$$

$$= 0$$

best approximation of second order

$$\therefore \int_a^b [f'' - S_f'']^2 dx \le \int_a^b [f'' - s'']^2 dx$$

12. $$Y(x) = \frac{a_0 + a_2 x^2 + a_4 x^4}{1 + b_2 x^2}$$ Padé approximation

Do Taylor expansion of $Y(x)$ at $x = 0$

$$\frac{1}{1+b_2x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot b_2^n \cdot x^{2n}$$

$$\frac{a_0+a_2x^2+a_4x^4}{1+b_2x^2} = (a_0+a_2x^2+a_4x^4)(1-b_2x^2+b_2^2x^4-b_2^3x^6+b_2^4x^8+\cdots)$$

$$= a_0+(a_2-a_0b_2)x^2+(a_0b_2^2-a_2b_2+a_4)x^4+(-a_0b_2^3+a_2b_2^2-a_4b_2)x^6+\cdots$$

$$\begin{cases} a_0 = 1 \\ a_2 - a_0b_2 = -\frac{1}{2!} \\ a_0b_2^2 - a_2b_2 + a_4 = \frac{1}{4!} \\ -a_0b_2^3 + a_2b_2^2 - a_4b_2 = \frac{-1}{6!} \end{cases}$$

Solve this nonlinear system

$$\begin{cases} a_0 = 1 \\ a_2 = -\frac{7}{15} \\ a_4 = \frac{1}{40} \\ b_2 = \frac{1}{30} \end{cases}$$

9.8. Using Peano kernel formula, we can get the exact formula of quadrature error

For midpoint rule.

$$\int_a^b f(x)dx - (b-a)\cdot f\left(\frac{a+b}{2}\right) = \int_a^b f''(t)\, K_0(t)\,dt \qquad K_0(t) = \begin{cases} \frac{1}{2}(t-a)^2 & t \le \frac{a+b}{2} \\ \frac{1}{2}(b-t)^2 & t > \frac{a+b}{2} \end{cases}$$

For trapezoid rule

$$\int_a^b f(x)dx - (b-a)\cdot\frac{f(a)+f(b)}{2} = \int_a^b f''(t)\, K_1(t)\,dt \qquad K_1(t) = \frac{1}{2}(a-t)(b-t)$$

check that $\int_a^b |K_0(t)|\, dt = \frac{1}{24}(b-a)^3$ $\qquad \int_a^b |K_1(t)|\,dt = \frac{1}{12}(b-a)^3$

$\therefore\ |E_0(f)| \le \|f''\|_\infty \cdot \frac{1}{24} h^3.$ $\qquad |E_1(f)| \le \|f''\|_\infty \cdot \frac{1}{12} h^3$

So $|E_1(f)| \simeq 2|E_0(f)|$ especally when $f(t)$ is quadratic function $f''(t)$ is constant, $|E_1(f)| \equiv 2|E_0(f)|$

9.2 midpoint open formulae $n=0$

So $M_0 = \int_{-1}^{1} t\,\pi_1(t)\,dt = \int_{-1}^{1} t\cdot t\,dt = \frac{2}{3}$

trapezoid rule. closed formulae $n=1$

$$k_1 = \int_0^1 \pi_2(t)\,dt = \int_0^1 t(t-1)\,dt = \frac{-1}{6}$$

Cavalieri–Simpson formula. closed formula $n=2$

$$M_2 = \int_0^2 t\,\pi_3(t)\,dt = \int_0^2 t^2(t-1)(t-2)\,dt = \frac{-4}{15}$$

9.3 $I_2(f) = \frac{2}{3}\left[2f(-\tfrac{1}{2}) - f(0) + 2f(\tfrac{1}{2})\right]$

$\int_{-1}^{1} 1\cdot dx = 2 = I_2(1)$ $\int_{-1}^{1} x\,dx = 0 = I_2(x)$

$\int_{-1}^{1} x^2 dx = \frac{2}{3} = I_2(x^2)$ $\int_{-1}^{1} x^3 dx = 0 = I_2(x^3)$

$\int_{-1}^{1} x^4 dx = \frac{2}{5} \neq I_2(x^4) = \frac{1}{6}$

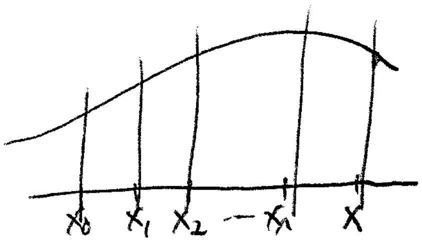
open formula. $n=2$, order 5

$I_4(f) = \frac{1}{4}\left[f(-1) + 3f(-\tfrac{1}{3}) + 3f(\tfrac{1}{3}) + f(1)\right]$

check the integration for $I_4(f)$ to get the degree of exactness

closed formula. $n = 3$ order ~~degree~~ 5

9.4. $\frac{d}{dx} f[x_0 x_1 x_2 \cdots x_n, x] = f[x_0 x_1 \cdots x_n, x, x]$



Consider x distinct from $\{x_i\}_{i=0}^{n}$

denote the unique n degree polynomial $P_0(t)$ that interpolates on $\{x_i\}_{i=0}^{n}$

then the unique $(n+1)$ degree polynomial $P_1(t)$ that interpolates on $\{x_0 x_1 \cdots x_n, x\}$

is $P_1(t) = P_0(t) + f[x_0 x_1 \cdots x_n, x]\cdot(t-x_0)(t-x_1)\cdots(t-x_n)$

$= P_0(t) + f[x_0 x_1 \cdots x_n, x]\, W_{n+1}(t)$

$\therefore f[x_0 x_1 \cdots x_n, x] = \dfrac{f(x) - P_0(x)}{W_{n+1}(x)}$ $\qquad \dfrac{d}{dx} f[x_0 x_1 \cdots x_n, x] = \dfrac{[f'(x) - P_0'(x)]W_{n+1} - [f(x) - P_0(x)]W_{n+1}'}{W_{n+1}^2(x)}$

The unique $(n+2)$ degree polynomial that interpolates on $\{x_0 x_1 x_2 \cdots x_n, x, x\}$ is

$P_2(t) = P_0(t) + f[x_0 x_1 \cdots x_n, x]\, W_{n+1}(t) + f[x_0 x_1 \cdots x_n, x, x]\cdot W_{n+1}(t)\,(t-x)$

So $f'(x) = P_0'(x) + f[x_0 x_1 \cdots x_n, x]\, W_{n+1}'(x) + f[x_0 x_1 \cdots x_n, x, x]\cdot\left(\cancel{W_{n+1}'(x)(x-x)} + W_{n+1}(x)\right)$

$\therefore f[x_0 x_1 \cdots x_n, x, x] = \dfrac{f(x) - P_0'(x) - f[x_0 \cdots x_n, x]\cdot W_{n+1}'(x)}{W_{n+1}(x)} = \dfrac{d}{dx} f[x_0 \cdots x_n, x]$