Math 270B: Numerical Approximation and Nonlinear Equations

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Homework Assignment #3

Due : Give to the class TA within a couple of weeks if you would like it marked before the final.

This homework covers the main material in 270B related to **approximation theory**. *Our goal is to study four basic problems in approximation theory: interpolation of real-valued functions of a single real variable using polynomials, best approximation, numerical differentiation, and quadrature (numerical approximation of integrals). We will again use many of the basic ideas about vector spaces and linear operators that we studied in the first homework, together with our background from 270A.*

Exercise 3.1. (Polynomial interpolation.)

Construct the (unique) interpolating polynomial of degree 3 for $\log x$ on the interval $[1, 4]$ using equi-spaced points (including the endpoints of the interval). Now construct an approximation to the derivative of $\log x$ using the interpolating polynomial you just constructed. Finally, construct an approximation to $\int_1^4 \log x dx$ using again your interpolating polynomial.

Exercise 3.2. (Automating interpolation with a computer.)

Construct a simple program (e.g. MATLAB or C) which builds the divided difference table described in class, given as input the sampling points on the real line $x_0, \ldots, x_n$, and the value of a real-valued function at those points, $f(x_0), \ldots, f(x_n)$.

As output from this program, return just the top diagonal of the table, namely the values $f[x_0]$, $f[x_0, x_1], \ldots, f[x_0, \ldots, x_n]$. (Recall that we only need these entries to build the Newton form of the interpolating polynomial.)

Also, write a second simple program, which uses the output from your divided difference program, to evaluate the interpolating polynomial at any input point x:

$$p_n(x) = \sum_{k=0}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j).$$

Exercise 3.3. (Using your automated interpolator.)

Build the interpolation polynomial of degree n, using $n+1$ equally spaced points in $I = [0, 1]$, $x_i = i/n$, $i = 0, 1, \ldots, n$, for the following function:

$$f(x) = \begin{cases} -1, & x \in [0, 1/2) \\ 1, & x \in [1/2, 1] \end{cases}$$

To build the polynomial, use your program above, with $n = 10, 12, \ldots, 20$.

What happens near $x = 1/2$ as n increases? Graph a few of the interpolating polynomials.

Exercise 3.4. (Best L^p-approximation.)

Consider the problem of best L^p-approximation of a (continuous) function $u(x)$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$: Find $u^* \in V$ such that

$$\|u - u^*\|_{L^p} = \min_{v \in V} \|u - v\|_{L^p},$$

where

$$\|u\|_{L^p} = \left(\int_0^1 |u|^p \, dx \right)^{1/p}, \ 1 \le p < \infty, \qquad \|u\|_{L^\infty} = \sup_{x \in [0,1]} |u(x)|.$$

Consider now the specific case of $u(x) = x^2$.

1. Determine the best L^2-approximation in the subspace of constant functions; i.e., $V = \operatorname{span}\{1\}$.
2. Determine the best L^2-approximation in the subspace of linear functions; i.e., $V = \operatorname{span}\{1, x\}$.
3. Determine the best L^∞-approximation in the subspace of linear functions. (This one is more challenging.).

Exercise 3.5. (Best approximation in abstract spaces.) In this problem, we will prove existence and uniqueness of best approximation, in the case of a Hilbert space H and a closed convex subset $M \subset H$. Prove the following:

Theorem. Let M be a closed convex subset of a Hilbert space H. Then for every $x \in H$, $\exists!\ y \in M$ such that

$$\|x - y\|_H = \inf_{w \in M} \|x - w\|_H \equiv \operatorname{dist}(x, M) \equiv d.$$

Part 1 of Proof. Given an argument justifying the assumption that there exists a sequence $\{y_n\}$, $y_n \in M$, such that

$$\lim_{n \mapsto \infty} \|x - y_n\|_H = \inf_{w \in M} \|x - w\|_H.$$

Hint for Part 1: Use the definition of the *inf.*

Part 2 of Proof. Argue that:

$$\|x - \frac{1}{2}(y_m + y_n)\|_H \geq d = \inf_{w \in M} \|x - w\|_H, \qquad \forall m, n \in \mathbb{N}.$$

Hint for Part 2: Use convexity of M.

Part 3 of Proof. Show that:

$$\|y_m - y_n\|_H^2 = 2\|x - y_m\|_H^2 + 2\|x - y_n\|_H^2 - 4\|x - \frac{1}{2}(y_m + y_n)\|_H^2.$$

Hint for Part 3: Use the Parallelogram Law (with the right choice of arguments). This is where having a Hilbert space leads to an easier proof than having only a Banach space.

Part 4 of Proof. Prove that the minimizing sequence $\{y_n\}$ is in fact Cauchy, that it converges to a $y \in M$, and that:

$$\|x - y\|_H = d.$$

Note that this $y \in M$ is the best approximation that we were looking for, but we don't know if it is unique.

Hint for Part 4: Use Parts 1, 2, 3, and a property of the norm to show the Cauchy property for $\{y_n\}$. Then use what you know about H and M to establish the existence of the limit point $y \in M$, and finally, use the continuity of the norm (proof given in class) to establish the inequality above.

Part 5 of Proof. Prove that this best approximation $y \in M$ is in fact unique.

Hint for Part 5: Use the standard technique of assuming there are two best approximations y and $\bar{y}$, and then show they must be the same. The key trick in showing this is to consider $\frac{1}{2}(y + \bar{y}) \in M$, and then use the Parallelogram Law again, along with some properties of the norm.

Exercise 3.6. (Finite Differences.)

- Determine the approximation error in the following difference formulae for an analytic function $f(x)$:
  1. $f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$
  2. $f''(x) \approx \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2}$
- Derive the second approximation above by differentiating an interpolating polynomial.

Exercise 3.7. (Quadrature.)

- Show that $\int_a^b \psi_2(x)dx = 0$ if

$$\psi_2(x) = (x - a)(x - \frac{a+b}{2})(x - b).$$

- Compare the Midpoint, Trapezoid, and Simpson Rules (non-composite) for approximating:

$$\int_0^1 x e^{-x^2} dx$$

  Derive bounds on the error in each case and compare to the true error..
- Write a simple MATLAB program to apply the composite Simpson's Rule using n points.
- Determine the number of integration intervals n needed for the compose Simpson Rule to approximate the following integral to six digits of accuracy:

$$\int_0^{10} e^{-x^2}$$

- Apply your implementation to approximate the above integral; verify you see the accuracy predicted.

Exercise 3.8. (Gaussian Quadrature.)

- Let $P_k(x)$, $k = 0, 1, \ldots$ be an orthgonal family of polynomials, and let $x_0, x_1, \ldots, x_k$ be the $k+1$ distinct zeros of $P_{k+1}(x)$. Prove that the Lagrange polynomials:

$$L_{k,i}(x) = \prod_{j=0, j\neq i}^{k} \frac{(x - x_j)}{(x_i - x_j)}, \qquad i = 0, \ldots, k$$

  are mutually orthogonal.
- Show that the Gauss weights

$$A_i = \int_a^b L_{k,i}(x) w(x) dx$$

  are always positive. *Hint: use the property of the Lagrange polynomials above.*
- Determine a quadrature rule of the form:

$$\int_{-1}^{1} f(x) dx \approx A_0 f(-\frac{1}{2}) + A_1 f(0) + A_2 f(\frac{1}{2})$$

  which is exact for all polynomials of degree 2 or less.

Exercise 3.9. (Quadrature – parting shot)

- Use a Gaussian quadrature rule to approximate the following integral:

$$\int_0^1 e^{-x^2} dx.$$

  Hint: Use the coordinate transformation $x = [(b-a)t + (b+a)]/2$ *to change the interval from arbitrary* $[a, b]$ *to* $[-1, 1]$*, and then use a familiar family which is orthgonal on* $[-1, 1]$*.*
- Use your Simpson's rule implementation (earlier problem) to approximate the following integral to within 10^{-6}:

$$\int_0^\infty \frac{1}{1 + x^4} dx.$$

  Hint: Use the coordinate transformation $t = 1/x$ *to convert the improper integral to a proper integral. You will first have to write the original integral as a sum of two integrals.*

Exercise 3.10. (Piecewise polynomial approximation theory – Challenging.)

Let $u(x)$ be analytic on $[a, b]$, and consider a subdivision of the interval as $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ using $n + 1$ nodal points. We will not assume uniformity, so that the points may be placed as desired in the interval (but they must remain distinct). Define $h = max_i h_i$, where $h_i = x_i - x_{i-1}, i = 1, \ldots, n + 1$.

- Precisely define the $n+1$ continuous piecewise linear Lagrange functions $\phi_i(x)$ as follows. There are $n+1$ "hat" functions satisfying the Lagrange property at the $n + 1$ nodal points:

$$\phi_i(x_j) = \delta_{ij}, \qquad i = 0, \ldots, n.$$

  After you have defined (given explicit formulae) the hat functions, sketch a picture of a few neighboring functions.
- Let u_I denote the continuous piecewise linear polynomial interpolant of $u(x)$ on $[a, b]$, defined using the basis above. Prove the following fundamental L^2-error estimate for the continous piecewise linear interpolant:

$$\|u - u_I\|_{L^2} \leq Ch^2 \|u\|_{H^2}, \text{ where } \|u\|_{H^k} = \left(\int_a^b \sum_{i=0}^{k} \left| \frac{d^i u}{dx^i} \right|^2 dx \right)^{1/2}.$$

- Prove the following fundamental H^1-error estimate:

$$\|u - u_I\|_{H^1} \leq Ch \|u\|_{H^2}.$$

- Prove the following L^∞ error estimate:

$$\|u - u_I\|_{L^\infty} \leq Ch^2 \|u\|_{H^2}.$$

Exercise 3.11. (General Approximation Theory – Challenging) Give a proof (or at least a fairly completely outline of the proof) of the Weierstrass Approximation Theorem.