Solution Keys of MATH 270A

Question 1:

$\boxed{2.3}$ (a) Since $A = A^*$ $\Rightarrow$ x^*Ax and x^*x are hermitian, where x is eigenvector of eigenvalue λ.

Consider $x^*Ax = (x^*Ax)^* = \lambda x^*x$, both sides are real.

(b) Use the fact from (a)

Consider
$$\begin{aligned} y^*Ax &= \lambda_1 y^*x \\ x^*Ay &= \lambda_2 x^*y \end{aligned}$$
, actually they are equal.

$\boxed{2.5}$ (a) Similarly as 2.3 (a).

(b) If $\lambda = 0$ is an eigenvalue of $I - S$

$\Rightarrow$ $(I-S)x = \lambda x$ $\Rightarrow$ $S^*x = (1-\lambda)x$

$\Rightarrow$ $1-\lambda$ i.e 1 is an eigenvalue of S.

Contradiction.

(c) Try to show $[(I-S)^{-1}(I+S)]^{-1} = [(I-S)^{-1}(I+S)]^*$

by using $S^* = -S$.

$\boxed{2.6}$ (a). Let $AA^{-1} = I$ $\Rightarrow$ $I + (1+\alpha+\alpha v^*u)uv^*$

must be zero.

(b) Consider Null (A) that contains nonzero element.

$\boxed{3.2}$ Use the defination of Matrix induced norm

$\boxed{3.3}$ (c) and (d) can be derived by applying (a) and (b).

$\boxed{3.4}$ (b) Hint: To show that the "deletion matrix" has norm less than "1".

$\boxed{4.4}$ Yes. Let $A = \cancel{\alpha}\, U_1 \Sigma V_1^*$, $B = U_2 \Sigma V_2^*$, notice that for two unitary matrice ~~A and B~~ U_1 and U_2, $U_1 \cdot U_2$ is still unitary.

$\boxed{4.5}$ Hints: Since $A^* = A^T$, notice that $A^* A$ is a square matrix. and $A^*A = V \Sigma^* \Sigma V^*$, then apply ~~the (4)~~ the facts
$= V \Sigma^* \Sigma V^{-1}$
that real square matrix has real singular values, and looks the product column by column.

Question 2.

$\boxed{6.1}$ Use The 6.1.

$\boxed{6.3}$ Use the fact that $Rank(A) = Rank(A^*A)$

$\boxed{6.5}$ Pick $v \in Range(P)$, then $Pv = v \Rightarrow \|P\|_2 \geq 1$
For equality, use facts $P^* = P$ and defination of $\|\cdot\|_2$.

$\boxed{7.5}$ Use facts:

i) $\text{Rank}(AB) \le \min(\text{Rank} A, \text{Rank } B)$

ii) if $B \in \mathbb{R}^{n \times k}$ has full rank

$\text{Rank}(AB) = \text{rank}(A)$.

$\boxed{11.1}$ Let $A = U\Sigma V^*$, try to simplify A^+'s SVD s.t. we can see $\|A^+\|_2 = \frac{1}{\sigma_n}$, where σ_n is the smallest singular value of A.

Question 3.

$\boxed{20.1}$ To show the uniqueness, you can look at the eigenvalues of A by ~~assume~~ assuming there $\exists$ another pair of $\bar{L}$ and $\bar{U}$.

$\boxed{20.2}$ Can be shown by ~~using the~~ observing the way we construct L.

$\boxed{20.3}$ (b) Same as 20.2, try to show it by construct the matrix L

$\boxed{21.2}$ Same as 20.2, try to construct P and L.

$\boxed{21.6}$ show that the resulting matrix ~~opt~~ after each step of Gaussian elimination is still diagonally dominant.

$\boxed{22.1}$ Since $U = L_1 \cdots L_{m-1} \cdot PA$, where $\|P\|_\infty = 1$

Consider $u_i = a_i - \sum_{j=1}^{i-1} l_{ij} u_j$, where $|l_{ij}| \le 1$.

To show the result by induction.

$\boxed{23.1}$ ~~No~~ Yes, because A^*A is a Hermite with positive eigenvalues.

Question 4.

$\boxed{12.1}$ Notice that $\|\cdot\|_F = \sqrt{\sum \sigma_i^2}$, where σ_i are singular values.

and $\|\cdot\|_2 = \sigma_{max}$.

$\boxed{14.1}$ For example:

(a) $\sin x = O(1) \iff \exists$ a constant C

s.t. $|\sin x| \le C$ with sufficent large x.

which is ~~correct~~ True.

$\boxed{14.2}$ (a) $(1+O(\varepsilon))(1+O(\varepsilon))$

$$= 1 + 2O(\varepsilon) + O(\varepsilon)^2 = 1 + O(\varepsilon).$$

$\boxed{16.1}$ To show it is backward stable, that is to show

for a $\tilde{A}$ with $\dfrac{\|A-\tilde{A}\|}{\|A\|} = O(\varepsilon)$, we have.

$\tilde{B} = Q_k \cdots Q_1 \tilde{A}$, ~~i.e. $\tilde{B} = Q_k Q \cdots Q_1 (A+\delta A)$~~

from right to left

i.e. for $\tilde{A} = A + \delta A$

$$Q_1(A+\delta A) = Q_1 A + Q_1 \delta A$$

$$Q_2 \cdot Q_1(A+\delta A) = Q_2 Q_1 A + Q_2 Q_1 \delta A$$

$$\vdots$$

$$\tilde{B} = Q_k \cdots Q_1 A + Q_k \cdots Q_1 \delta A = Q_k \cdots Q_1 (A+\delta A).$$

$\boxed{17.1}$ (a) $\|\delta R\|_1 \le \max_i \sum_j |\delta r_{ij}| \le m \max_i \sum_j |r_{ij}| \, \varepsilon + O(\varepsilon^2)$

$\boxed{17.2}$ $\|\tilde{x} - x\|$, where $\begin{cases} (R+\delta R)\tilde{x} = b \\ Rx = b \end{cases} \Rightarrow R\tilde{x} - Rx = -\delta R\tilde{x}$

$\Rightarrow R(x-\tilde{x}) = \delta R\tilde{x} = \delta R(R+\delta R)^{-1}b$ . . .

$\boxed{18.4}$ For square case, y can be obtained exactly independent of A.

$\boxed{19.1}$ (19.4) is $\begin{cases} r + Ax = b \\ A^* r = 0 \end{cases}$, since least square solution solves $A^*Ax = A^*b$.

One can show x from the system is the same in $A^*Ax = A^*b$.

Question 5.

$\boxed{24.4}$ (a) $A^n = U[\lambda^n \ldots]U^*$ . . .

$\boxed{25.1}$ (a) By hint, ~~the~~ then look at the dimension of eigenspace.

$\boxed{25.2}$ (a) To ~~rek~~ reach accuracy $O(\varepsilon)$ we need iterate m times where $c^m = O(\varepsilon)$

$\Rightarrow m = \frac{\log \varepsilon}{\log c}$, totally $m \cdot O(1) = O(\log \varepsilon)$.

$\boxed{26.3}$ (b). Use the fact A is normal

$\iff$ A is diagonalizable by unitary matrix.

then apply Bauer-Fike result.

$\boxed{27.1}$ For a x, consider $\frac{x}{\|x\|}$ be the i-th column of Q.