We have

$$\hat{M}_-\hat{M}_+ = (\hat{M}_x - i\hat{M}_y)(\hat{M}_x + i\hat{M}_y)$$
$$= \hat{M}_x^2 + \hat{M}_y^2 + i[\hat{M}_x, \hat{M}_y] = \hat{M}^2 - \hat{M}_z^2 - \hbar\hat{M}_z. \qquad (7.10.37)$$

If the operator $\hat{M}_-\hat{M}_+$ acts on $|\lambda, m_{\max}\rangle$, it follows by using (7.10.31) that

$$(\hat{M}^2 - \hat{M}_z^2 - \hbar\hat{M}_z)|\lambda, m_{\max}\rangle = \hat{M}_-\hat{M}_+|\lambda, m_{\max}\rangle = 0. \qquad (7.10.38)$$

Therefore, by (7.10.26) and (7.10.27),

$$(\lambda - m_{\max}^2 - m_{\max})|\lambda, m_{\max}\rangle = 0$$

or

$$\lambda = m_{\max}^2 + m_{\max}. \qquad (7.10.39)$$

Similarly,

$$\hat{M}_+\hat{M}_- = \hat{M}^2 - \hat{M}_z^2 + \hbar\hat{M}_z. \qquad (7.10.40)$$

If this operator acts on $|\lambda, m_{\min}\rangle$, and (7.10.34) is used, we obtain

$$\lambda - m_{\min}^2 + m_{\min} = 0. \qquad (7.10.41)$$

If we equate the two results for λ from (7.10.39) and (7.10.41), it turns out that

$$(m_{\max} + m_{\min})(m_{\min} - m_{\max} - 1) = 0 \qquad (7.10.42)$$

Thus

$$m_{\max} = -m_{\min}. \qquad (7.10.43)$$

Therefore the admissible values of m lie symmetrically about the origin. Since the extreme values differ by an integer, it follows that

$$m_{\max} - m_{\min} = 2l, \qquad (7.10.44)$$

where

$$l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots. \qquad (7.10.45)$$

These results combined with (7.10.43) show that

$$-l \le m \le l \qquad (2l+1 \text{ values}). \qquad (7.10.46)$$

Finally, it follows from (7.10.39) and (7.10.44) that

$$\lambda = l(l+1), \qquad l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots. \qquad (7.10.47)$$

This is a definite proof for integer and half-integer eigenvalues for the angular momentum. Particles with integral spin are called the *Bosons*, those with half-integral spins are known as *Fermions*.

The two different kinds of angular momentum operators can be combined to define the *total angular momentum*

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{M}} \qquad (7.10.48)$$

with the components $\hat{J}_x = \hat{L}_x + \hat{M}_x$, $\hat{J}_y = \hat{L}_y + \hat{M}_y$, $\hat{J}_z = \hat{L}_z + \hat{M}_z$.

It follows from the properties of $\hat{L}$ and $\hat{M}$ that $\hat{J}$ satisfies the usual commutation relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z, \qquad (7.10.49a)$$
$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, \qquad (7.10.49b)$$
$$[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y, \qquad (7.10.49c)$$

and hence

$$[\hat{J}_x, \hat{\mathbf{J}}^2] = [\hat{J}_y, \hat{\mathbf{J}}^2] = [\hat{J}_z, \hat{\mathbf{J}}^2] = 0, \qquad (7.10.50abc)$$

where

$$\hat{\mathbf{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2. \qquad (7.10.51)$$

It can readily be shown that

$$\hat{\mathbf{J}}^2|l, m\rangle = l(l+1)\hbar^2|l, m\rangle, \qquad (7.10.52)$$
$$\hat{J}_z|l, m\rangle = \hbar m|l, m\rangle. \qquad (7.10.53)$$

This means that the eigenvalues of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$ are $l(l+1)\hbar^2$ and $\hbar m$, respectively, where $|m| \le l$ and the quantum numbers may be either integers or half-integers.

Finally, it follows that

$$[\hat{J}^2, \hat{L}_z] = 2\hat{M}_x[\hat{L}_x, \hat{L}_z] + 2\hat{M}_y[\hat{L}_y, \hat{L}_z] = \hbar[\hat{L}_+, \hat{L}_-], \qquad (7.10.54)$$
$$[\hat{\mathbf{J}}^2, \hat{M}_z] = -\hbar[\hat{L}_+, \hat{L}_-]. \qquad (7.10.55)$$

7.11. Exercises

(1) (a) Use the Lagrangian, $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2)$ for the three dimensional isotropic harmonic oscillator, and Lagrange's equations of motion to show that the total energy is constant where k is the force constant.

(b) Show that the Lagrangian for the oscillator in spherical polar coordinates (r, θ, ϕ) is

$$L = T - V = \tfrac{1}{2}m(r^2 + r^2\dot{\theta}^2 + r^2 \sin^2\phi\, \dot{\phi}^2) - \tfrac{1}{2}kr^2,$$

where $k = 4\pi^2 m\omega^2$.

Hence write down the Lagrange equations of motion.

(2) Consider a single particle of mass m moving in a plane under a conservative force with potential $V(r)$, where r is distance from the origin of coordinates. With r and θ as generalized coordinates describing the motion of the particle, show that the corresponding momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

where $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$. Hence show that

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r), \qquad mr^2\dot{\theta} = \text{constant}, \; m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r}.$$

Give an interpretation of each of the above results.

(3) If A is a complex dynamical function of q and p, A^* is its complex conjugate, and if the Poisson bracket $\{A, A^*\} = i$, compute $\{A, AA^*\}$, $\{A, A^*A\}$, $\{A^*, AA^*\}$ and $\{A^*, A^*A\}$.

(4) Find the Hamiltonian and Hamilton's equations of motion for

(i) The simple harmonic oscillator, $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$ and
(ii) The planetary motions, $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, and $V = m\mu(1/2a - 1/r)$. In this case, derive the differential equations for the central orbit.

(5) Establish the following results for the Poisson brackets:

(i) $\{A, B\} = -\{B, A\}$,
(ii) $\{(A+B), C\} = \{A, C\} + \{B, C\}$,
(iii) $\{AB, C\} = \{A, C\}B + A\{B, C\}$,
(iv) $\{A, \alpha\} = 0$,
(v) $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$ (Jacobi's Identity),

where A, B, C are canonical functions and α is a scalar.

(6) Show that

(i) $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$,
(ii) $[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$,
(iii) $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$,
(iv) $[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$,
(v) $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$,
(vi) $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$ (Jacobi's Identity),
(vii) $[\hat{A}^2, \hat{B}] = \hat{A}[\hat{A}, \hat{B}] + [\hat{A}, \hat{B}]\hat{A}$,
(viii) $[\hat{A}, \alpha] = 0$, α is a scalar.

(7) For the three dimensional position and momentum operators of a particle, prove that

$$[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij},$$

where the suffixes i, j take the values 1, 2, 3 for the x, y, z components of $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$, respectively.

(8) By direct evaluation for canonically conjugate variables q and p, show that

(i) $[p^2, q^2] = 2\hbar^2 - 4i\hbar pq$,
(ii) $[p, q^2] = -2i\hbar q$,
(iii) $[\hat{x}^2, \hat{p}_x^2] = 2\hbar^2 - 4i\hat{x}\hat{p}_x$,
(iv) $[\hat{p}_x, \hat{x}^2] = -2i\hbar\hat{x}$.

(9) If A and B are any two operators which both commute with their commutator $[\hat{A}, \hat{B}]$ prove that

$$[\hat{A}, \hat{B}^n] = n\hat{B}^{n-1}[\hat{A}, \hat{B}],$$

$$[\hat{A}^n, \hat{B}] = n\hat{A}^{n-1}[\hat{A}, \hat{B}].$$

(10) Establish the following commutator relations:

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0.$$

(11) Show that

$$[\hat{L}_+, \hat{L}_-] = \hbar\hat{L}_z,$$

$$[\hat{L}_+, \hat{L}_z] = -\hbar\hat{L}_+,$$

$$[\hat{L}_-, \hat{L}_z] = \hbar\hat{L}_-,$$

$$[\hat{L}_x, \hat{L}_+] = \hbar\hat{L}_-,$$

$$[\hat{L}^2, \hat{L}_+] = 0.$$

(12) Prove that

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2 + \hat{\mathbf{M}}^2 + 2\hat{\mathbf{L}} \cdot \hat{\mathbf{M}} = \hat{\mathbf{L}}^2 + \hat{\mathbf{M}}^2 + 2\hat{L}_z\hat{M}_z + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+$$

$$2\hat{\mathbf{L}} \cdot \hat{\mathbf{M}} = \hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{M}}^2.$$

(13) Show that the probability for a position measurement on the state $\Psi(x, t)$ to yield a value somewhere between x_1 and x_2 is

$$P(x_1, x_2, t) = \int_{x_1}^{x_2} \bar{\Psi}\Psi\, dx = \int_{x_1}^{x_2} |\Psi(x, t)|^2\, dx.$$

Using the Schrödinger equations, derive the result

$$\frac{d}{dt} P(x_1, x_2, t) = J(x_1, t) - J(x_2, t),$$

where

$$J(x, t) = \frac{i\hbar}{2m}\left[\Psi\frac{\partial\bar{\Psi}}{\partial x} - \bar{\Psi}\frac{\partial\Psi}{\partial x}\right].$$

(14) Use the inner product

$$(\phi, \psi) = \int_{-\infty}^{\infty} \bar{\phi}\psi\, dx,$$

and the property $(\phi, \psi) \to (0, 0)$ as $|x| \to \infty$, to show that the position operator $\hat{x} = x$, the momentum operator $\hat{p} = -i\hbar\, \partial/\partial x$, and the energy operator $\hat{H} = \hat{p}^2/2m + \hat{V}(\hat{x})$ are Hermitian operators.

(15) Establish the following commutation relations for the orbital angular momentum operators:

$$[\hat{L}_x, \hat{x}] = 0,$$

$$[\hat{L}_x, \hat{y}] = i\hbar\hat{z},$$

$$[\hat{L}_x, \hat{z}] = -i\hbar\hat{y},$$

$$[\hat{L}_x, \hat{p}_x] = 0,$$

$$[\hat{L}_x, \hat{p}_y] = i\hbar\hat{p}_z,$$

$$[\hat{L}_x, \hat{p}_z] = -i\hbar\hat{p}_y.$$

(16) Prove the Heisenberg uncertainty relation for the harmonic oscillator

$$\Delta x\, \Delta p \geq \frac{1}{2}\hbar.$$

(17) If $\hat{A}$ and $\hat{B}$ are constants of motion, show that the commutator $i[\hat{A}, \hat{B}]$ is also a constant of motion.

(18) Show that, for the linear harmonic oscillator,

$$[\hat{H}, \hat{A}] = (-\hbar\omega)\hat{A}, \qquad [\hat{H}, \hat{A}^*] = (\hbar\omega)\hat{A}^*,$$

where

$$\hat{A} = \hat{a}/\sqrt{\hbar\omega} \quad \text{and} \quad \hat{A}^* = \hat{a}^*/\sqrt{\hbar\omega}.$$

(19) For the three dimensional anisotropic Planck's oscillator, the Hamiltonian is given by

$$H_r = \frac{1}{2m} p_r^2 + \frac{1}{2} m\omega_r^2 x_r^2, \qquad r = 1, 2, 3,$$

so that total Hamiltonian $H = H_1 + H_2 + H_3$ and the total energy $E = E_1 + E_2 + E_3$, where E_1, E_2, E_3 are energies of each of the independent degrees of freedom. Show that

$$E = (n_1 + \tfrac{1}{2})\hbar\omega_1 + (n_2 + \tfrac{1}{2})\hbar\omega_2 + (n_3 + \tfrac{1}{2})\hbar\omega_3.$$

In the case of an isotropic oscillator, $\omega_1 = \omega_2 = \omega_3 = \omega$, derive the result

$$E_N = (N + \tfrac{3}{2})\hbar\omega, \qquad N = n_1 + n_2 + n_3 = 0, 1, 2, 3, \ldots$$

(20) Prove the compatibility theorem which states that any one of the following conditions implies the other two:

(i) $\hat{A}$ and $\hat{B}$ are compatible,
(ii) $\hat{A}$ and $\hat{B}$ possess a common eigenbasis,
(iii) $\hat{A}$ and $\hat{B}$ commute,

where A and B are two observables with corresponding operators $\hat{A}$ and $\hat{B}$.

(21) If the eigenvectors $\{\psi_n(x)\}$ form an orthonormal basis in a Hilbert space, show that any state vector $\psi(x)$ satisfies the result

$$(\psi, \psi) = \sum_{n=1}^{\infty} |(\psi_n, \psi)|^2.$$

(22) If $\hat{A}' \equiv \hat{A} - \langle\hat{A}\rangle$ and $\hat{B}' \equiv \hat{B} - \langle\hat{B}\rangle$, prove the following results:

(i) $\hat{A}'$ and $\hat{B}'$ are Hermitian operators,
(ii) $[\hat{A}', \hat{B}'] = [\hat{A}, \hat{B}]$,
(iii) $(\hat{A}'\psi, \hat{A}'\psi) = (\Delta\hat{A})^2$.

Use these results to establish the generalized uncertainty relation.

(23) Using $\langle\hat{A}\rangle = \int_{-\infty}^{\infty} \Psi^*(x)[\hat{A}\Psi(x)]\, dx$ prove that the expectation values of position and momentum in the state $\Psi(x, t)$ are

$$\langle\hat{x}\rangle = \int_{-\infty}^{\infty} x|\Psi(x, t)|^2\, dx, \qquad \langle\hat{p}\rangle = -i\hbar\int_{-\infty}^{\infty} \Psi^* \frac{\partial}{\partial x}\Psi(x, t)\, dx.$$

Also show that

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi(x, t)|^2 \, dx, \qquad \langle \hat{p}^2 \rangle = \hbar^2 \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} \Psi(x, t) \right|^2 dx.$$

(24) Apply the basic commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ and rules of commutator algebra to show that

$$[\hat{x}\hat{p}_x, \hat{H}] = \frac{i\hbar}{m} \hat{p}_x^2 + \hat{x}[\hat{p}_x, \hat{V}], \qquad [\hat{y}\hat{p}_y, \hat{H}] = \frac{i\hbar}{m} \hat{p}_y^2 + \hat{y}[\hat{p}_y, \hat{V}],$$

$$[\hat{z}\hat{p}_z, \hat{H}] = \frac{i\hbar}{m} \hat{p}_z^2 + \hat{z}[\hat{p}_z, \hat{V}].$$

Hence combine them to obtain the Heisenberg equation of motion for the operator $\mathbf{r} \cdot \mathbf{p}$

$$\frac{d}{dt} \langle \mathbf{r} \cdot \mathbf{p} \rangle = \left\langle \frac{\mathbf{p}^2}{m} \right\rangle - \langle \mathbf{r} \cdot \nabla V \rangle.$$

Hence or otherwise prove the *Virial Theorem* for the stationary states:

$$2\langle T \rangle = \langle \mathbf{r} \cdot \nabla V \rangle.$$

(25) Use the results in Exercise (9) for $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}_x$ to prove that for any Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \alpha \hat{x}^n,$$

the following relation holds:

$$[\hat{x}\hat{p}_x, \hat{H}] = i\hbar \left(\frac{\hat{p}_x^2}{m} - \alpha n \hat{x}^n \right) = i\hbar(2\hat{T} - n\hat{V}).$$

(26) Use the Hamiltonian operator for the one dimensional simple harmonic oscillator in the form

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m\omega^2 \hat{x}^2,$$

and then introduce the non-dimensional variables

$$\hat{X} = \left(\frac{m\omega}{2\hbar} \right)^{1/2} \hat{x}, \qquad \hat{P} = \frac{1}{(2m\hbar\omega)^{1/2}} \hat{p}.$$

(a) Show that

(i) $\hat{X}$ and $\hat{P}$ are Hermitian operators,
(ii) $\hat{H} = \hbar\omega(\hat{P}^2 + \hat{X}^2)$,
(iii) $[\hat{X}, \hat{P}] = \frac{1}{2}i$.

(b) If $\hat{Q} = \hat{X} + i\hat{P}$ and $\hat{Q}^* = \hat{X} - i\hat{P}$, show that

$$\hat{Q}\hat{Q}^* = \hat{X}^2 + \hat{P}^2 + \tfrac{1}{2}, \qquad \hat{Q}^*\hat{Q} = \hat{X}^2 + \hat{P}^2 - \tfrac{1}{2}, \qquad \hat{H} = \hbar\omega(\hat{Q}^*\hat{Q} + \tfrac{1}{2}),$$

where the algebra of the operators $\hat{Q}$ and $\hat{Q}^*$ is defined by the commutation relation

$$[\hat{Q}, \hat{Q}^*] = 1.$$

(27) If $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are two vector operators that commute with the Pauli spin matrices but do not commute between themselves, prove the Dirac identity

$$(\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{A}})(\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{B}}) = (\hat{\mathbf{A}} \cdot \hat{\mathbf{B}}) + \mathrm{i}(\hat{\mathbf{A}} \times \hat{\mathbf{B}}) \cdot \hat{\boldsymbol{\sigma}},$$

where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$.