Math 210A: Mathematical Physics

Instructor: Michael Holst

Fall Quarter 2015

Homework Assignment #1
Due (Give to Prof. Holst a couple of week before final if you want feedback)

Exercise 1.1. (*Useful Facts to Derive*)

- Let $p > 1$, $q > 1$, and $1/p + 1/q = 1$. Prove that for every $0 \le x \le 1$,

$$x^{1/p} \le \frac{x}{p} + \frac{1}{q}.$$

  Hint: This is Exercise 1.9(5) in 1st Ed. (1.7(8) in 3rd Ed.)
- Use the fact above to prove Young's inequality: For $a, b \ge 0$, $1 < p, q < \infty$, $1/p + 1/q = 1$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

  Hint: This is proved in the book.
- Do Exercise 1.9(15) in 1st Ed. (1.7(18) in 3rd Ed.)

Exercise 1.2. (*Vector Spaces and Subspaces*)

- Do Exercise 1.9(4) in 1st Ed. (1.7(4) in 3rd Ed.)
- Do Exercise 1.9(6) in 1st Ed. (1.7(9) in 3rd Ed.)
- Do Exercise 1.9(7) in 1st Ed. (1.7(10) in 3rd Ed.)
- Do Exercise 1.9(9) in 1st Ed. (1.7(12) in 3rd Ed.)
- Do Exercise 1.9(11) in 1st Ed. (1.7(14) in 3rd Ed.)
- Do Exercise 1.9(13) in 1st Ed. (1.7(16) in 3rd Ed.)
- Do Exercise 1.9(39) in 1st Ed. (1.7(50) in 3rd Ed.)

Exercise 1.3. (*Normed Spaces as Metric Spaces*)
Let $\|\cdot\| : X \to \mathbb{R}$ be a norm on a vector space X with associated scalar field $\mathbb{R}$. We know that $\|\cdot\|$ must satisfy the three properties of a norm:

1. $\|u\| \ge 0, \quad \forall u \in X, \quad \|u\| = 0$ iff $u = 0$.
2. $\|\alpha u\| = |\alpha|\|u\|, \quad \forall \alpha \in \mathbb{R}, \quad \forall u \in X.$
3. $\|u + v\| \le \|u\| + \|v\|, \quad \forall u, v \in X.$

Use these properties to show the induced metric $d(u, v) = \|u - v\|$ satisfies the three properties of a metric:

1. $d(u, v) \ge 0, \quad \forall u, v \in X, \quad d(u, v) = 0$ iff $u = v$.
2. $d(u, v) = d(v, u), \quad \forall u, v \in X.$
3. $d(u, v) \le d(u, w) + d(w, v), \quad \forall u, v \in X.$

Hint: This was proved in lecture.

Exercise 1.4. (*Inner-Product Spaces as Normed Spaces*)
Let $(\cdot,\cdot) : X \times X \to \mathbb{R}$ be an inner-product on a vector space X with associated scalar field $\mathbb{R}$. We know that $(\cdot,\cdot)$ must satisfy the three properties of an inner-product:

1. $(u, u) \geq 0, \quad \forall u \in X, \quad (u, u) = 0$ iff $u = 0$.
2. $(u, v) = (v, u), \quad \forall u, v \in X$.
3. $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w), \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall u, v, w \in X$.

Use these three properties to show the induced norm $\|u\| = (u, u)^{1/2}$ satisfies the three norm properties.
Hint: Showing the first two properties is very easy, in fact we did it in lecture. To show the last property (triangle inequality), assume the Cauchy-Schwarz inequality holds (or even better, prove it): $|(u, v)| \leq \|u\|\|v\|, \quad \forall u, v \in X$.

Exercise 1.5. (*Equivalent Norms*)
Let X be a normed space. Recall that two norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ on X are called *equivalent* if:

$$C_1\|u\|_X \leq \|u\|_Y \leq C_2\|u\|_X, \quad \forall u \in X.$$

In the case of a finite-dimensionals space X, we noted that all norms can be shown to be equivalent. Consider now the specific finite-dimensional space $X = \mathbb{R}^n$.

1. Determine constants C_1 and C_2 for the l^p norms for $p = 1, 2, \infty$. In particular, show the following tight bounds:
$$\|u\|_\infty \leq \|u\|_2 \leq \|u\|_1 \leq \sqrt{n}\|u\|_2 \leq n\|u\|_\infty, \quad \forall u \in \mathbb{R}^n. \tag{1.1}$$
2. Use these inequalities to show that if $A \in \mathbb{R}^{n\times n}$, then the corresponding matrix norms also have equivalence relationships:
$$\|A\|_1 \leq \sqrt{n}\|A\|_2 \leq n\|A\|_1, \tag{1.2}$$
$$\|A\|_\infty \leq \sqrt{n}\|A\|_2 \leq n\|A\|_\infty. \tag{1.3}$$
3. Derive the analogous equivanence relationships for the corresonding condition numbers (assume now that A is inverible).
4. Show that for any norm $\|\cdot\|$ on $\mathbb{R}^n$, if $\rho(A)$ is the spectral radius of $A \in \mathbb{R}^{n\times n}$, then
$$\rho(A) \leq \|A\|.$$

Hint: This is based on Exercise 1.9(25) in 1st Ed. (Does not appear in quite this form in 3rd Ed.)

Exercise 1.6. (*Convergent and Cauchy Sequences, Completeness*)

- Do Exercise 1.9(23) in 1st Ed. (1.7(28) in 3rd Ed.)
- Do Exercise 1.9(28) in 1st Ed. (1.7(33) in 3rd Ed.)
- Do Exercise 1.9(31) in 1st Ed. (1.7(40) in 3rd Ed.)
- Do Exercise 1.9(32) in 1st Ed. (1.7(41) in 3rd Ed.)
- Do Exercise 1.9(33) in 1st Ed. (1.7(42) in 3rd Ed.)

Exercise 1.7. (*Banach Spaces*)

- Do Exercise 1.9(34) in 1st Ed. (1.7(43) in 3rd Ed.)
- Do Exercise 1.9(35) in 1st Ed. (1.7(44) in 3rd Ed.)

Exercise 1.8. (*Linear Operators on Normed Spaces*)

- Do Exercise 1.9(36) in 1st Ed. (1.7(45) in 3rd Ed.)
- Do Exercise 1.9(38) in 1st Ed. (1.7(47) in 3rd Ed.)

Exercise 1.9. (*The Banach Fixed-Point Theorem*)

- Do Exercise 1.9(40) in 1st Ed. (1.7(52) in 3rd Ed.)
- Do Exercise 1.9(41) in 1st Ed. (1.7(53) in 3rd Ed.)
- Do Exercise 1.9(42) in 1st Ed. (1.7(55) in 3rd Ed.)