MATH 210A: Mathematical Physics

Instructor: Michael Holst

Fall Quarter 2015

Final Exam

Due Date: Please get this back to me by the end of finals week.

Problem 1. (*Vector Spaces and Subspaces*) Show that the monomials x^n, $n = 0, 1, 2, \ldots$ are linearly independent.

Problem 2. (*Inner-product Spaces, Normed Spaces, Metric Spaces*)
Let $(\cdot, \cdot) : X \times X \to \mathbb{R}$ be an inner-product on a vector space X with associated scalar field $\mathbb{R}$. Show that the induced norm $\|u\| = (u, u)^{1/2}$ satisfies the three norm properties, and show that the metric induced by the induced norm $d(u, v) = \|u - v\|$ satisfies the three properties of a metric.

Problem 3. (*Convergent and Cauchy Sequences, Completeness, Banach Spaces*) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in a normed space X. Show that the sequence must also be a Cauchy sequence. When do we know that the reverse is true? Give an example of a normed space that is a Banach space, i.e., a normed space where all Cauchy sequences converge.

Problem 4. (*Orthonormal Systems, Linear Operators, Hilbert Spaces*) Let $a(u, v)$ be a bounded and coercive bilinear form on a Hilbert space X, and let $f(v)$ be a bounded linear functional on X. Use the theorems we learned in class to show that the problem:

$$\text{Find } u \in X \text{ such that } a(u, v) = f(v), \quad \forall v \in X,$$

is equivalent to the problem:

$$\text{Find } u \in X \text{ such that } Au = F \in X,$$

where $A \in \mathcal{L}(X, X)$.

Problem 5. (*The Banach Fixed-Point Theorem*) Consider the linear system $Au = F$ from the previous problem under the given assumptions.

1. Derive (show all steps) an equivalent fixed-point formulation

   $$\text{Find } u \in X \text{ such that } u = G(u),$$

   where $G(u) = [I - \rho A]u + \rho f$, $\rho \in \mathbb{R}$, $\rho \neq 0$.
2. Now derive conditions on ρ that will guarantee that G is a contraction, so that you can conclude the original problem has a unique solution through use of the Banach Fixed Point Theorem.
3. Finally, derive an *a priori* bound on the solution of the form:

   $$\|u\|_X \leq C,$$

   where C is a function of the boundedness and coercivity constants arising in the assumptions on the bilinear form and linear functional. This last step allows us to finally conclude that the original problem in the previous step is well-posed. (And, you have also just proven the Lax-Milgram Theorem.)

Problem 6. (*Calculus in Banach Spaces, Calculus of Variations*) Let X and Y be real Hilbert spaces, let $A \in \mathcal{L}(X, Y)$, $f \in Y$, and define $J \colon X \to \mathbb{R}$ as:

$$J(u) = \frac{1}{2}\|f - Au\|_X^2.$$

Show that a necessary condition for $J(u)$ to have a local minimizer $u \in X$ is

$$0 = J'(u) = A^T Au - A^T f,$$

where $A^T \in \mathcal{L}(Y, X)$ is the Hilbert-adjoint of A.

Problem 7. (*Calculus in Banach Spaces, Calculus of Variations*) Let X be a real Hilbert space, let $A \in \mathcal{L}(X,X)$, $f \in Y$, and define $J\colon X \to \mathbb{R}$ as:

$$J(u) = \frac{1}{2}(Au, u) - (f, u).$$

Show that a necessary condition for $J(u)$ to have a local minimizer $u \in X$ is

$$0 = J'(u) = \frac{1}{2}(A + A^T)u - f,$$

where $A^T \in \mathcal{L}(X,X)$ is the Hilbert-adjoint of A. Write a simpler form of the condition when A is self-adjoint.

Problem 8. (*Taylor Expansion in Banach Spaces, Newton's Method*) Let X be a real Hilbert space, let $F\colon X \to X$, and consider the problem:

$$\text{Find } u \in X \text{ such that } F(u) = 0 \in X.$$

1. Using Taylor expansion, derive a simple Newton method (with uniform step-size $\alpha = 1$) for solving this problem:
   - Choose arbitrary $u^0 \in X$, choose $TOL << 1$.
   - For $k = 0, 1, 2, \ldots$ until $(\|F(u^k)\|_X < TOL)$ do:
     (a) Solve: $F'(u^k)p^k = -F(u^k)$,
     (b) Update: $u^{k+1} = u^k + \alpha p^k$.
   - End For.
2. Recall that a *direction of decrease* $p \in X$ for a functional $J\colon X \to \mathbb{R}$ at the point $u \in X$ satisfies

   $$J(u + \alpha p) < J(u), \quad \forall \alpha \in (0, \sigma], \text{ with } \sigma \text{ sufficiently small.}$$

   Recall that a *descent direction* $p \in X$ for a functional $J\colon X \to \mathbb{R}$ at the point $u \in X$ satisfies

   $$(J'(u), p) < 0.$$

   Use a simple one-dimensional Taylor expansion to show that a descent direction is always a direction of decrease.
3. Now show that the Newton direction p is actually a descent direction (hence a direction of decrease) for the special choice of J:

   $$J_F(u) = \frac{1}{2}\|F(u)\|_X^2 = \frac{1}{2}((F(u), F(u))_X.$$

   To show this, you will need to compute the Gateaux derivative of $J_F(u)$ (show your work):

   $$J_F'(u) = F'(u)^T F(u).$$

4. Use this result to design a *Damped Newton Iteration* that produces iteratives u^k that are guaranteed (through control of steplength α) to reduce the value of the special functional J_F at each step. (This is sometimes called "globalizing Newton iteration"; the algorithm you have produced is at the core of nearly all of computational science.)

Problem 9. (*Hilbert Space Tools Applied to a PDE Problem*)

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be open and convex with a smooth boundary $\partial\Omega$. Consider the elliptic boundary value problem: Find $u\colon \Omega \to \mathbb{R}$ such that

$$-\nabla \cdot (a\nabla u) + bu = f, \quad \text{in } \Omega, \tag{1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

where $a, b, f\colon \Omega \to \mathbb{R}$, where $f \in L^2(\Omega)$, and where

$$0 < a_0 \leq a(x) \leq a_1 < \infty, \qquad 0 \leq b(x) \leq b_1 < \infty, \qquad \forall x \in \overline{\Omega}. \tag{3}$$

1. Derive a mathematically equivalent weak formulation that involves looking for a solution $u \in H_0^1(\Omega)$.
2. Show that the bilinear form you produce is bounded and coercive, and that the linear functional you produce is bounded. (*Hint:* You will use the assumptions on a, b, f, together with the Poincare inequality.)
3. Apply the Lax-Milgram Theorem to conclude that this problem is well-posed. For the *a priori* norm bound on u, show explicitly where the assumptions on the coefficients appear in the bound.