An operator A is said to have a *compact-normal resolvent* if there exists a scalar λ such that $(\lambda \mathcal{I} - A)^{-1}$ is a compact and normal operator. To apply the above theorem we need to determine whether a given operator A has a compact-normal resolvent.

We close this section with the following, rather interesting, remark.

Let A be a closed operator in a Hilbert space H. We know that this does not imply boundedness of A. On the other hand, it is always possible to redefine the inner product on $\mathcal{D}(A)$ such that $\mathcal{D}(A)$ becomes a Hilbert space and A becomes a bounded operator on $\mathcal{D}(A)$. In fact, for $x, y \in \mathcal{D}(A)$ define

$$(x, y)_1 = (x, y) + (Ax, Ay),$$

where $(\cdot, \cdot)$ denotes the inner product in H. The proof of completeness of $\mathcal{D}(A)$ with respect to the norm

$$\|x\|_1 = \|x\|^2 + \|Ax\|^2,$$

and the boundedness of A in this new Hilbert space is left as an exercise.

4.13. Exercises

(1) If A is an operator on H such that $Ax \perp x$ for every $x \in H$, show that $A = 0$.

(2) Let A be a bounded operator defined on a proper subspace of a Hilbert space H.

(a) Define an operator A_1 on the closure $\overline{\mathcal{D}(A)}$ of the domain of A by

$$A_1 x = \lim_{n\to\infty} Ax_n, \qquad \text{where } x_n \in \mathcal{D}(A) \text{ and } x_n \to x.$$

Show that A_1 is well defined, i.e., $A_1 x$ does not depend on a particular choice of the sequence $\{x_n\}$. Show that A_1 is a linear and bounded operator defined on $\overline{\mathcal{D}(A)}$.

(b) Define an operator B on H by

$$Bx = A_1 x_1, \qquad \text{where } x_1 \text{ is the projection of } x \text{ onto } \overline{\mathcal{D}(A)}.$$

Show that B is a bounded operator on H.

(c) Show that $\|A\| = \|B\|$.

Since $A = B$ on $\mathcal{D}(A)$, B is an extension of A.

(3) Let ϕ be a symmetric, positive, bilinear functional on a vector space E. Show that

$$|\phi(x, y)|^2 \le \phi(x, x)\phi(y, y).$$

(4) Let $\{e_n\}$ be a complete orthonormal sequence in a Hilbert space H and let $\{\lambda_n\}$ be a sequence of scalars.

(a) Show that there exists a unique operator T on E such that $Te_n = \lambda_n e_n$.
(b) Show that T is bounded if and only if the sequence $\{\lambda_n\}$ is bounded.
(c) For a bounded sequence $\{\lambda_n\}$, find the norm of T.

(5) Let $A: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $A[x, y] = [x + 2y, 3x + 2y]$. Find the eigenvalues and eigenvectors of A.

(6) Let $T: \mathbf{C}^2 \to \mathbf{C}^2$ be defined by $T[x, y] = [x + 3y, 2x + y]$. Show that $T^* \neq T$.

(7) Let $A: \mathbf{R}^3 \to \mathbf{R}^3$ be given by $A[x, y, z] = [3x - z, 2y, -x + 3z]$. Show that A is self-adjoint.

(8) Compute the adjoint of each of the following operators:
(a) $A: \mathbf{R}^3 \to \mathbf{R}^3$, $A[x, y, z] = [-y + z, -x + 2z, x + 2y]$,
(b) $B: \mathbf{R}^3 \to \mathbf{R}^3$, $B[x, y, z] = [x + y - z, -x + 2y + 2z, x + 2y + 3z]$,
(c) $C: \mathcal{P}_2(\mathbf{R}) \to \mathcal{P}_2(\mathbf{R})$, $C(p(x)) = x(d/dx)(p(x)) - (d/dx)(xp(x))$,

where $\mathcal{P}_2(\mathbf{R})$ is the space of all polynomials on $\mathbf{R}$ of degree less than or equal to 2.

(9) If A is a self-adjoint operator and B is a bounded operator, show that B^*AB is self-adjoint.

(10) Prove that the representation $T = A + iB$ in Theorem 4.4.4 is unique.

(11) If $A^*A + B^*B = 0$, show that $A = B = 0$.

(12) Let A be an operator on H. Show that

(a) A is anti-Hermitian if and only if iA is self-adjoint.
(b) $A - A^*$ is anti-Hermitian.

(13) Show that if T is self-adjoint and $T \neq 0$, then $T^n \neq 0$ for all $n \in \mathbf{N}$.

(14) Let A be a self-adjoint operator. Show that

(a) $\|Ax + ix\|^2 = \|Ax\|^2 + \|x\|^2$,
(b) The operator $U = (A - i\mathcal{I})(A + i\mathcal{I})^{-1}$ is unitary. (U is called the *Cayley transform* of A.)

(15) The limit of a convergent sequence of self-adjoint operators is a self-adjoint operator.

(16) If T is a bounded operator on H with one dimensional range, show that there exist vectors $y, z \in H$ such that $Tx = (x, z)y$ for all $x \in H$. Hence show that

(a) $T^*x = (x, y)z$ for all $x \in H$,
(b) $T^2 = \lambda T$, λ is a scalar,
(c) $\|T\| = \|y\| \|z\|$,
(d) $T^* = T$ if and only if $y = \alpha z$ for some real scalar α.

(17) Let A be a bounded self-adjoint operator on a Hilbert space H such that $\|A\| < 1$. Prove that $(x, Tx) \geq (1 - \|A\|)\|x\|^2$ for all $x \in H$.

(18) Show that the product of isometric operators is an isometric operator.

(19) Let $\{e_n\}$ be a complete orthonormal sequence in a Hilbert space H. Show that an operator A on H is unitary if and only if $\{Ae_n\}$ is a complete orthonormal sequence in H.

(20) Let $\{e_n\}$, $n \in \mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, be a complete orthonormal system in a Hilbert space H. Show that there exists a unique operator A on H such that $Ae_n = e_{n+1}$ for all $n \in \mathbf{Z}$. Operator A is called a *two-sided shift operator.* Show that A is isometric and unitary.

(21) Show that the product of two unitary operators is a unitary operator.

(22) Let A be an operator on a Hilbert space. Define the exponential operator by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \qquad (0^0 = \mathscr{I}).$$

Show that e^A is a well-defined operator. Prove the following

(a) $(e^A)^n = e^{nA}$ for any $n \in \mathbf{N}$,
(b) $e^0 = \mathscr{I}$,
(c) e^A is invertible (even if A is not) and its inverse is e^{-A},
(d) $e^A e^B = e^{A+B}$ for any commuting operators A and B,
(e) If A is self-adjoint, then e^{iA} is unitary.

(23) If T is a normal operator on H and λ is a scalar, show that

$$\|T^*x - \bar{\lambda}x\| = \|Tx - \lambda x\| \qquad \text{for all } x \in H.$$

(24) Show that if the kernel $K(x, y)$ satisfies $K(x, y) = \overline{K(y, x)}$, then for any real α the operator

$$(Tu)(x) = \alpha u(x) + i \int_a^b K(x, y) u(y)\, dy$$

on $L^2([a, b])$ is normal.

(25) Show that for any invertible operator T, the operator T^*T is also invertible.

(26) If T is normal, show that T is invertible if and only if T^*T is invertible.

(27) Prove Theorem 4.5.4.

(28) Let T and S be commuting operators. Show that if both T and S are normal, then $S + T$ and ST are normal.

(29) If $T^*T = \mathscr{I}$, is it true that $TT^* = \mathscr{I}$?

(30) Let A, B, C, and D be positive operators on a Hilbert space. Prove the following

(a) If $A \geq B$ and $C \geq D$, then $A + C \geq B + D$.
(b) If $A \geq 0$ and $\alpha \geq 0$ ($\alpha \in \mathbf{R}$), then $\alpha A \geq 0$.
(c) If $A \geq B$ and $B \geq C$, then $A \geq C$.
(d) If $A \geq 0$ and $\|A\| \leq 1$, then $A \leq \mathscr{I}$.
(e) If $A \geq 0$, then there exists $\alpha > 0$ ($\alpha \in \mathbf{R}$) such that $\alpha A \leq \mathscr{I}$.

(31) If A is a positive operator and B is a bounded operator, show that B^*AB is positive.

(32) If A and B are positive operators and $A + B = 0$, show that $A = B = 0$.

(33) Show that for any self-adjoint operator A there exist positive operators S and T such that $A = S - T$ and $ST = 0$.

(34) If A is a positive definite operator, then it is invertible and its inverse is positive definite.

(35) Find operators $T: \mathbf{R}^2 \to \mathbf{R}^2$ such that $T^2 = \mathscr{I}$. Which one is the positive square root of $\mathscr{I}$?

(36) Find the positive square root of the operator T on $L^2([a, b])$ defined by $(Tf)(t) = g(t)f(t)$, where g is a positive continuous function on $[a, b]$.

(37) Show that $\|\sqrt{A}\| = \sqrt{\|A\|}$.

(38) Let A and B be positive operators on a Hilbert space. Show that $A^2 = B^2$ implies $A = B$.

(39) Let A and B be commuting positive operators. Show that $\sqrt{AB} = \sqrt{A}\sqrt{B}$.

(40) If P is self-adjoint and P^2 is a projection operator, is P a projection operator?

(41) Let T be a multiplication operator on $L^2([a, b])$. Find necessary and sufficient conditions for T to be a projection.

(42) Give an example of two non-commuting projection operators.

(43) Show that P is a projection if and only if $P = P^*P$.

(44) Generalize Theorem 4.7.3 to any finite sum of projections.

(45) Show that every projection P is a positive operator and $0 \leq P \leq \mathcal{I}$.

(46) If T is an isometric operator, show that TT^* is projection.

(47) Show that for projections P and Q the operator $P+Q-PQ$ is a projection if and only if $PQ = QP$.

(48) Prove Theorem 4.8.2.

(49) Show that the projection onto a closed subspace F of a Hilbert space H is a compact operator if and only if F is finite dimensional.

(50) Show that the operator $T: l^2 \to l^2$ defined by $T(\{x_n\}) = \{2^{-n}x_n\}$ is compact.

(51) Show that a self-adjoint operator T is compact if and only if there exists a sequence of finite dimensional operators strongly convergent to T.

(52) Prove that the collection of all eigenvectors corresponding to one particular eigenvalue of an operator is a vector space.

(53) Show that the space of all eigenvectors corresponding to one particular eigenvalue of a compact operator is finite dimensional.

(54) Show that eigenvalues of a symmetric operator are real and eigenvectors corresponding to different eigenvalues are orthogonal.

(55) Show that every non-zero vector is an eigenvector of the operator $A = \alpha\mathcal{I}$ corresponding to the eigenvalue α.

(56) Show that shift operators have no eigenvalues.

(57) Give an example of a self-adjoint operator which has no eigenvalues.

(58) Give an example of a normal operator which has no eigenvalues.

(59) Show that a non-zero vector x is an eigenvalue of an operator A if and only if $|(Ax, x)| = \|Ax\| \|x\|$.

(60) Show that if the eigenvectors of a self-adjoint operator A form a complete orthogonal system and all eigenvalues are non-negative (or positive) then A is positive (or strictly positive).

(61) Prove the Spectral Theorem for the finite dimensional case: If $T: \mathbf{R}^N \to \mathbf{R}^N$ is a self-adjoint operator, then there exists an orthonormal system of vectors $\phi_1, \ldots, \phi_N \in \mathbf{R}^N$ and scalars $\lambda_1, \ldots, \lambda_N \in \mathbf{C}$ such that

$$T\phi_k = \lambda_k \phi_k, \qquad k = 1, \ldots, N.$$

Hence the matrix corresponding to T relative to the basis $\{\phi_1, \ldots, \phi_N\}$ is

$$\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & \ldots & \lambda_N \end{bmatrix}.$$

(62) If λ is an approximate eigenvalue of an operator T, show that $|\lambda| \leq \|T\|$.

(63) Show that if T has an approximate eigenvalue λ such that $|\lambda| = \|T\|$, then $\sup_{\|x\| \leq 1} |(Tx, x)| = \|T\|$.

(64) If λ is an approximate eigenvalue of T, show that $\lambda + \mu$ is an approximate eigenvalue of $T + \mu\mathcal{I}$ and $\lambda\mu$ is an approximate eigenvalue of μT.

(65) Show that $|\lambda| = 1$ for every approximate eigenvalue λ of an isometric operator.

(66) Show that every approximate eigenvalue of a self-adjoint operator is real.

(67) Show that if λ is an approximate eigenvalue of a normal operator T, then $\bar{\lambda}$ is an approximate eigenvalue of T^*.

(68) Provide a detailed proof for Corollary 4.11.2.

(69) Prove Theorem 4.11.5.

(70) Find the Fourier transform of

(a)
$$f(x)=\begin{cases}1 & \text{if } x\in[-a,a],\\ 0 & \text{otherwise.}\end{cases}$$

(b)
$$f(x)=\begin{cases}1-|x|/2 & \text{if } x\in[-2,2],\\ 0 & \text{otherwise.}\end{cases}$$

(71) Use Example 4.11.1(b) and Theorem 4.11.5(c) to show that

$$\mathcal{F}\{e^{-a^2x^2}\}=\frac{1}{\sqrt{2}|a|}e^{-k^2/4a^2}.$$

(72) Show that under appropriate conditions

(a) $\hat{f}'(k)=-i\mathcal{F}\{xf(x)\}$.
(b) $\hat{f}^{(r)}(k)=(-i)^r\mathcal{F}\{x^rf(x)\}$.

(73) Use the Parseval relation to evaluate

(a)
$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2 dx,$$

(b)
$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^3 dx,$$

(c)
$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^4 dx.$$

(74) Prove that $(AB)C=A(BC)$ holds for unbounded operators.

(75) Prove that

(a) $(A+B)C=AC+BC$,
(b) $AB+AC\subset A(B+C)$,

holds for unbounded operators. Give an example of operators A, B, C for which $AB+AC\neq A(B+C)$.

(76) Show that $(A+B)^*\supset A^*+B^*$.

(77) Give an example of a closed operator whose domain is not a closed set.

(78) Show that A^{**} is symmetric whenever A is symmetric.

(79) If A is an operator on a Hilbert space H and there exists an operator B on H such that $(Ax, y)=(x, By)$ for all $x, y\in H$, show that A is bounded and $B=A^*$.

(80) Let A be a closed operator in a Hilbert space H. Prove that $\mathcal{D}(A)$ is a Hilbert space with respect to the inner product defined by

$$(x, y)_1=(x, y)+(Ax, Ay),$$

where $(\cdot,\cdot)$ denotes the inner product in H. Prove that A is a bounded operator on $\mathcal{D}(A)$ with the defined inner product.