$x, y \in S_1$ we have

$$\begin{aligned}\|x-y\|^2 &= (x-y, x-y)\\ &= (x,x)-(x,y)-(y,x)+(y,y)\\ &= 1-0-0+1 \qquad \text{(by the orthogonality)}\\ &= 2.\end{aligned}$$

This means that the distance between any two distinct elements of S_1 is $\sqrt{2}$.

Now consider the collection of $\frac{1}{2}\sqrt{2}$-neighborhoods about every element of S_1. Clearly, no two of these neighborhoods can have a common point. Since every dense subset of H must have at least one point in every neighborhood and H has a countable dense subset, S_1 has to be countable. Thus, S is countable, proving the theorem.

Definition 3.12.2 (Hilbert Space Isomorphism). A Hilbert space H_1 is said to be *isomorphic* to a Hilbert space H_2 if there exists a one-to-one linear mapping T from H_1 onto H_2 such that

$$(T(x), T(y)) = (x, y) \tag{3.12.1}$$

for every $x, y \in H_1$. Such a mapping T is called a *Hilbert space isomorphism* of H_1 onto H_2.

Note that (3.12.1) implies $\|T\| = 1$, because $\|T(x)\| = \|x\|$ for every $x \in H_1$.

Theorem 3.12.3. *Let H be a separable Hilbert space.*

(a) *If H is infinite dimensional, then it is isomorphic to l^2;*
(b) *If H has a dimension N, then it is isomorphic to $\mathbf{C}^N$.*

Proof. Let $\{x_n\}$ be a complete orthonormal sequence in H. If H is infinite dimensional, then $\{x_n\}$ is an infinite sequence. Let x be an element of H. Define $T(x) = (\alpha_1, \alpha_2, \ldots)$, where $\alpha_n = (x, x_n)$, $n = 1, 2, \ldots$ By Theorem 3.8.3, T is a one-to-one mapping from H onto l^2. It is clearly a linear mapping. Moreover, for $\alpha_n = (x, x_n)$ and $\beta_n = (y, x_n)$, $x, y \in H$, $n \in \mathbf{N}$, we have

$$\begin{aligned}(T(x), T(y)) &= ((\alpha_1, \alpha_2, \ldots), (\beta_1, \beta_2, \ldots))\\ &= \sum_{n=1}^{\infty} \alpha_n \overline{\beta_n} = \sum_{n=1}^{\infty} (x, x_n)\overline{(y, x_n)}\\ &= \sum_{n=1}^{\infty} (x, (y, x_n)x_n) = \left(x, \sum_{n=1}^{\infty} (y, x_n)x_n\right) = (x, y).\end{aligned}$$

Thus T is an isomorphism from H onto l^2.

The proof of (b) is left as an exercise.

Remarks. 1. It is easy to check that isomorphism of Hilbert spaces is an equivalence relation.

2. Since any infinite dimensional separable Hilbert space is isomorphic to l^2, it follows that any two such spaces are isomorphic. The same is true for real Hilbert spaces; any real infinite dimensional separable Hilbert space is isomorphic to the real space l^2. In some sense, there is only one real and one complex infinite dimensional separable Hilbert space.

3.13. Exercises

(1) Show that

$$(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z) \qquad \text{for all } \alpha, \beta \in \mathbf{C},$$

in any inner product space.

(2) Prove that the space $\mathscr{C}_0(\mathbf{R})$ of all complex valued continuous functions that vanish outside some finite interval is an inner product space with the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

(3) Verify that the spaces in Examples 3.3.1–3.3.7 are inner product spaces.

(4) (a) Let $E = \mathscr{C}^1([a, b])$ (the space of all continuously differentiable complex valued functions on $[a, b]$). For $f, g \in E$ define

$$(f, g) = \int_a^b f'(x)\overline{g'(x)}\, dx.$$

Is $(\cdot, \cdot)$ an inner product in E?

(b) Let $F = \{f \in \mathscr{C}^1([a, b]) : f(a) = 0\}$. Is $(\cdot, \cdot)$ defined in (a) an inner product in F?

(5) Is the space $\mathscr{C}_0^1(\mathbf{R})$ of all continuously differentiable complex valued continuous functions that vanish outside some finite interval an inner product space if

$$(f, g) = \int_{-\infty}^{\infty} f'(x)\overline{g'(x)}\, dx \ ?$$

(6) Show that the norm in an inner product space is *strictly convex*, i.e., if $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|x+y\| < 2$.

(7) Show that in any inner product space $\|x-y\|+\|y-z\|=\|x-z\|$ if and only if $y=\alpha x+(1-\alpha)z$ for some $\alpha\in[0,1]$.

(8) Let $E_1,\ldots,E_n$ be inner product spaces. Show that

$$([x_1,\ldots,x_n],[y_1,\ldots,y_n])=(x_1,y_1)+\cdots+(x_n,y_n)$$

defines an inner product in $E=E_1\times\cdots\times E_n$. If $E_1,\ldots,E_n$ are Hilbert spaces, show that E is a Hilbert space and its norm is defined by

$$\|[x_1,\ldots,x_n]\|=\sqrt{\|x_1\|^2+\cdots+\|x_n\|^2}.$$

(9) Show that the *polarization identity*

$$(x,y)=\frac{1}{4}[\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2]$$

holds in any pre-Hilbert space.

(10) Show that for any x in a Hilbert space $\|x\|=\sup_{\|y\|=1}|(x,y)|$.

(11) Prove that any complex Banach space with norm $\|\cdot\|$ satisfying the parallelogram law is a Hilbert space with the inner product defined by

$$(x,y)=\frac{1}{4}[\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2],$$

and then $\|x\|^2=(x,x)$.

(12) Is $\mathscr{C}([a,b])$ with the norm $\|f\|=\max_{[a,b]}|f(x)|$ an inner product space?

(13) Show that $L^2([a,b])$ is the only inner product space among the spaces $L^p([a,b])$.

(14) Show that for any elements in an inner product space,

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{x+y}{2}\right\|^2.$$

The equality is called *Apollonius' identity.*

(15) Prove that any finite dimensional inner product space is a Hilbert space.

(16) Let $F=\{f\in\mathscr{C}^1([a,b]):f(a)=0\}$ and

$$(f,g)=\int_a^b f'(x)\overline{g'(x)}\,dx.$$

Is E a Hilbert space?

(17) Is the space $\mathscr{C}_0^1(\mathbf{R})$ with the inner product

$$(f,g)=\int_{-\infty}^{\infty} f'(x)\overline{g'(x)}\,dx$$

a Hilbert space?

(18) Let E be an incomplete inner product space. Let H be the completion of E (see Section 1.7). Is it possible to extend the inner product from E onto H such that H would become a Hilbert space?

(19) Suppose $x_n\to x$ and $y_n\to y$ in a Hilbert space, and $\alpha_n\to\alpha$ in $\mathbf{C}$. Prove that

(a) $x_n+y_n\to x+y$,
(b) $\alpha_n x_n\to\alpha x$,
(c) $(x_n,y_n)\to(x,y)$,
(d) $\|x_n\|\to\|x\|$.

(20) Suppose $x_n\xrightarrow{w}x$ and $y_n\xrightarrow{w}y$ in a Hilbert space, and $\alpha_n\to\alpha$ in $\mathbf{C}$. Prove or give a counterexample:

(a) $x_n+y_n\xrightarrow{w}x+y$;
(b) $\alpha_n x_n\xrightarrow{w}\alpha x$;
(c) $(x_n,y_n)\to(x,y)$;
(d) $\|x_n\|\to\|x\|$;
(e) If $x_n=y_n$ for all $n\in\mathbf{N}$, then $x=y$.

(21) Show that in a finite dimensional Hilbert space weak convergence implies strong convergence.

(22) It is always possible to find a norm on an inner product space E which would define the weak convergence in E?

(23) If $\sum_{n=1}^{\infty}u_k=u$, show that $\sum_{n=1}^{\infty}(u_k,x)=(u,x)$ for any x in an inner product space.

(24) Let $\{x_1, \ldots, x_n\}$ be a finite orthonormal set in a Hilbert space H. Prove that for any $x \in H$ the vector

$$x - \sum_{k=1}^{n} (x, x_k) x_k$$

is orthogonal to x_k for every $k = 1, \ldots, n$.

(25) In the pre-Hilbert space $\mathscr{C}([-\pi, \pi])$ show that the following sequences of functions are orthogonal:

(a) $x_k(t) = \sin kt,\ k = 1, 2, 3, \ldots;$
(b) $y_n(t) = \cos nt,\ n = 0, 1, 2, \ldots.$

(26) Show that the application of the Gram-Schmidt process to the sequence of functions

$$f_0(t) = 1, f_1(t) = t, f_2(t) = t^2, \ldots, f_n(t) = t^n, \ldots$$

(as elements of $L^2([-1, 1])$) yields the Legendre polynomials.

(27) Show that the application of the Gram-Schmidt process to the sequence of functions

$$f_0(t) = e^{-t^2/2}, f_1(t) = t\, e^{-t^2/2}, f_2(t) = t^2\, e^{-t^2/2}, \ldots, f_n(t) = t^n\, e^{-t^2/2}, \ldots$$

(as elements of $L^2(\mathbf{R})$) yields the orthonormal system discussed in Example 3.7.4.

(28) Apply the Gram-Schmidt process to the sequence of functions

$$f_0(t) = 1, f_1(t) = t, f_2(t) = t^2, \ldots, f_n(t) = t^n, \ldots$$

defined on $\mathbf{R}$ with the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, e^{-t^2}\, dt.$$

Compare the result with Example 3.7.4.

(29) Apply the Gram-Schmidt process to the sequence of functions

$$f_0(t) = 1, f_1(t) = t, f_2(t) = t^2, \ldots, f_n(t) = t^n, \ldots$$

defined on $[0, \infty)$ with the inner product

$$(f, g) = \int_{0}^{\infty} f(t)\overline{g(t)}\, e^{-t}\, dt.$$

The obtained polynomials are called the *Laguerre polynomials*.

(30) Let T_n be the Chebyshev polynomials of degree n, i.e.,

$$T_0(x) = 1, \quad T_n(x) = 2^{1-n} \cos(n \arccos x).$$

Show that the functions

$$\phi_n(x) = \frac{2^n}{\sqrt{2\pi}} T_n(x), \qquad n = 0, 1, 2, \ldots,$$

form an orthonormal system in $L^2[(-1, 1)]$ with respect to the inner product

$$(f, g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) g(x)\, dx.$$

(31) Prove that for any polynomial

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

we have

$$\max_{[-1,1]} |p_n(x)| \geq \max_{[-1,1]} |T_n(x)|,$$

where T_n denotes the Chebyshev polynomial of degree n.

(32) Show that the complex functions

$$\phi_n(z) = \sqrt{\frac{n}{\pi}}\, z^{n-1}, \qquad n = 1, 2, 3, \ldots,$$

form an orthonormal system in the space of continuous complex functions defined in the unit disc $D = \{z \in \mathbf{C}: |z| \leq 1\}$ with respect to the inner product

$$(f, g) = \iint_D f(z)\overline{g(z)}\, dz.$$

(33) Prove that the complex functions

$$\psi_n(z) = \frac{1}{\sqrt{2\pi}}\, z^{n-1}, \qquad n = 1, 2, 3, \ldots,$$

form an orthonormal system in the space of continuous complex functions defined on the unit circle $C = \{z \in \mathbf{C}: |z| = 1\}$ with respect to the inner product

$$(f, g) = \int_C f(z)\overline{g(z)}\, dz.$$

(34) With respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\omega(x)\,dx,$$

where $\omega(x) = (1-x)^\alpha(1+x)^\beta$ and $\alpha, \beta > -1$, show that the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!2^n}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^\alpha(1+x)^\beta(1-x^2)^n]$$

form an orthogonal system.

(35) Show that the Gegenbauer polynomials

$$C_n^\gamma(x) = \frac{(-1)^n}{n!\,2^n}(1-x^2)^{1/2-\gamma}\frac{d^n}{dx^n}(1-x^2)^{n+\gamma-1/2},$$

where $\gamma > \frac{1}{2}$, form an orthonormal system with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{\gamma-1/2}\,dx.$$

Note that Gegenbauer polynomials are a special case of Jacobi polynomials with $\alpha = \beta = \gamma - \frac{1}{2}$.

(36) Find $a, b, c \in \mathbf{C}$ which minimize the value of the integral

$$\int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,dx.$$

(37) If x and x_k $(k = 1, \ldots, n)$ belong to a real Hilbert space, show that

$$\left\| x - \sum_{k=1}^{n} a_k x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} a_k(x, x_k) + \sum_{k=1}^{n}\sum_{s=1}^{n} a_k a_s(x_k, x_s).$$

Also show that this expression is minimum when $Aa = b$, where $a = (a_k)$, $b = ((x, x_k))$ and the matrix $A = (a_{ks})$ is defined by $a_{ks} = (x_k, x_s)$.

(38) If $\{a_n\}$ is an orthonormal sequence in a Hilbert space H and $\{\alpha_n\}$ is a sequence in l^2, show that there exists $x \in H$ such that

$$(x, a_n) = \alpha_n \quad \text{and} \quad \|\{\alpha_n\}\| = \|x\|,$$

where $\|\{\alpha_n\}\|$ denotes the norm in l^2.

(39) If a_n and b_n $(n = 1, 2, 3, \ldots)$ are generalized Fourier coefficients of vectors x and y with respect to a complete orthonormal sequence in a Hilbert space, show that

$$(x, y) = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

(40) Let $\{e_n\}$ be an orthonormal sequence in a Hilbert space H. Show that $\{e_n\}$ is complete if and only if $(x, y) = \sum_{n=1}^{\infty}(x, e_n)\overline{(y, e_n)}$ for every $x, y \in H$.

(41) Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H. Show that $\{x_n\}$ is complete if and only if $\text{cl}(\text{span}\{x_1, x_2, \ldots\}) = H$. In other words, $\{x_n\}$ is complete if and only if every element of H can be approximated by a sequence of finite combinations of x_n's.

(42) Show that the functions

$$\phi_n(x) = \frac{e^{-x/2}}{n!}L_n(x), \qquad n = 0, 1, 2, \ldots,$$

where L_n is the Laguerre polynomial of degree n, i.e.,

$$L_n(x) = e^x\frac{d^n}{dx^n}(x^n e^{-x}),$$

form a complete orthonormal system in $L^2([0, 1])$.

(43) Let

$$\phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

and let $f \in L^1([-\pi, \pi])$. Define

$$f_n(x) = \sum_{k=-n}^{n}(f, \phi_k)\phi_k,$$

for $n = 0, 1, 2, \ldots$. Show that

$$\frac{f_0(x) + f_1(x) + \cdots + f_n(x)}{n+1} = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)(f, \phi_k)\phi_k(x).$$

(44) Fill in the details of the proof of Lemma 3.9.1.

(45) Let f be a continuous non-negative function defined on $[-\pi, \pi]$ such that $\text{supp} f \subseteq [-\pi + \varepsilon, \pi - \varepsilon]$, for some $0 < \varepsilon < \pi$, and $\int_{-\pi}^{\pi} f(x)\,dx = 2\pi$. Define

$$g_n(x) = nf(n(x - \pi)) \text{ for } n = 1, 2, \cdots \text{ and } x \in [0, 2\pi].$$

Let k_n be the 2π-periodic extension of g_n onto the entire line $\mathbf{R}$. Show that $\{k_n\}$ is a summability kernel.

(46) Show that the sequence of functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots$$

is a complete orthonormal sequence in $L^2([-\pi, \pi])$.

(47) Show that the following sequence of functions is a complete orthonormal system in $L^2([0, \pi])$:

$$\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \cos x, \sqrt{\frac{2}{\pi}} \cos 2x, \sqrt{\frac{2}{\pi}} \cos 3x, \dots.$$

(48) Show that the following sequence of functions is a complete orthonormal system in $L^2([0, \pi])$:

$$\sqrt{\frac{2}{\pi}} \sin x, \sqrt{\frac{2}{\pi}} \sin 2x, \sqrt{\frac{2}{\pi}} \sin 3x, \dots.$$

(49) Give an example of a complete orthonormal sequence in $L^2([a, b])$ for arbitrary $a<b$.

(50) What is the orthogonal complement in $L^2(\mathbf{R})$ of the set of all even functions?

(51) What is the orthogonal complement in $L^2([-\pi, \pi])$ of the set of all polynomials of odd degree?

(52) Let $\mathscr{P}$ be a complete orthonormal system in a Hilbert space H. Show that if $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ and $\mathscr{P}_1 \cap \mathscr{P}_2 = \varnothing$, then $\mathscr{P}_1^\perp = \text{cl}(\text{span } \mathscr{P}_2)$.

(53) Let S be a subset of an inner product space. Show that $S^\perp = (\text{span } S)^\perp$.

(54) Let E be the Banach space $\mathbf{R}^2$ with the norm $\|(x, y)\| = \max\{|x|, |y|\}$. Show that E does not have the closest point property.

(55) Find a Banach space E, a closed convex subset $S \subset E$, and a point $x \notin S$, such that there is no $y \in E$ such that $\|x-y\| = \inf_{z \in S} \|x-z\|$.

(56) Let S be a closed subspace of a Hilbert space H and let $\{e_1, e_2, \dots\}$ be a complete orthonormal sequence in S. For an arbitrary $x \in H$ there exists $y \in S$ such that $\|x-y\| = \inf_{z \in S} \|x-z\|$. Define y in terms of $\{e_1, e_2, \dots\}$.

(57) If S is a closed subspace of a Hilbert space H, then $H = S \oplus S^\perp$. Is this true in every inner product space?

(58) Show that the functional in Example 3.11.2 is unbounded.

(59) The Riesz Representation Theorem says, that for every bounded linear functional $f \in H'$ on a Hilbert space H, there exists a representer $x_f \in H$ such that $f(x) = (x, x_f)$ for all $x \in H$. Let $T: H' \to H$ be the mapping which assigns x_f to f. Prove the following properties of T:

(a) T is onto,
(b) $T(f+g) = T(f) + T(g)$,
(c) $T(\alpha f) = \bar{\alpha} T(f)$,
(d) $\|T(f)\| = \|f\|$,

where $f, g \in H'$ and $\alpha \in \mathbf{C}$.

(60) Prove part (b) of Theorem 3.12.3.

(61) Let f be a bounded linear functional on a closed subspace F of a Hilbert space H. Show that there exists a bounded linear functional g on H such that $\|f\| = \|g\|$ and $f(x) = g(x)$ whenever $x \in F$.

(62) Show that the space l^2 is separable.

(63) Let $\mathscr{P}$ be an uncountable orthonormal system in an inner product space E. Show that for every $x \in E$ we have $(x, e) \neq 0$ for at most countably many $e \in \mathscr{P}$.