Hw 6 Sol^{n}

3.4 #8)

$$x' = x$$
$$y' = y + x^2$$

find an integral:

To proceed we use (3.15).

we find an integral of $(y+x^2)\,dx - x\,dy = 0$

This is not exact, so introduce an integrating factor $m(x)$.

then $\frac{\partial}{\partial y}\left(m(x)\,(y+x^2)\right) = \frac{\partial}{\partial x}\left(m(x)\,(-x)\right)$

$$m(x) = m'(x)(-x) - m(x)$$

$$2\,m(x) = (-x)\,m'(x)$$

$$\frac{m'(x)}{m(x)} = \frac{-2}{x}$$

$$\ln m(x) = \ln x^{-2}$$
$$m(x) = \frac{1}{x^2}$$

then for $\left(\frac{y}{x^2} + 1\right)dx - \frac{1}{x}\,dy = 0$

$$\int -\frac{1}{x}\,dy = -\frac{y}{x} = Q$$

$$F = Q + H(x)$$

$$P = \frac{\partial}{\partial x}F = \frac{\partial}{\partial x}\left(Q + H(x)\right) = \frac{y}{x^2} + H'(x) \quad \text{so} \quad H'(x) = 1$$

and $H(x) = x$

so $F = -\frac{y}{x} + x$ is an integral.

15) The chain rule gives

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}. \text{ If } \frac{dx}{dt} = 1, \text{ then}$$

$$\frac{dy}{dt} = \frac{dy}{dx}\left(\frac{dx}{dt}\right) = \frac{dy}{dx}(1) = \frac{dy}{dx} = f(x,z)$$

Thus the system is the same as the eq^{n} $\frac{dy}{dx} = f(x,y)$.

16) Similar to 15, so we write

$$\frac{dx}{dz} = f(x,y,z)$$

$$\frac{dy}{dz} = g(x,y,z).$$

Using the same trick, this system can be rewritten

$$\frac{dz}{dt} = 1$$

$$\frac{dx}{dt} = \frac{dx}{dz}\frac{dz}{dt} = \frac{dx}{dz}(1) = f(x,y,z)$$

$$\frac{dy}{dz} = \frac{dy}{dz}\left(\frac{dz}{dt}\right) = \frac{dy}{dz}(1) = g(x,y,z)$$

3.5)

#1: a) As the equation has a stable point at $(d, b/a)$, we should release b/a predators, which will keep pest population fixed.

b) This is a bad strategy. Reducing the number of pests will have the effect of moving the population away from the stable point. Regardless of the number of predators released, eventually the pest population will boom.

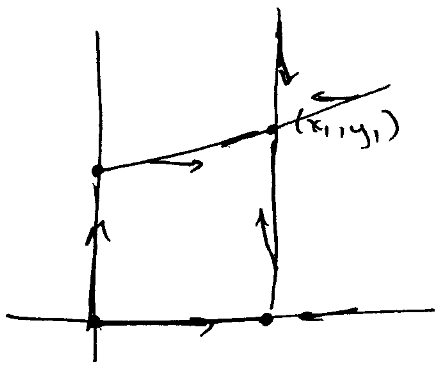
*1 c). The ideal strategy is to wait until the pest population approaches d, then release b/a predators.

5) x-nullclines at $x=0$ $\quad K-x+By=0$

y-nullclines at $y=0$ $\quad L+Cx-y=0$

then there are at most 4 stationary points in the case where $BC<1$,

the stationary point at the intersection of the two lines is given by $x_1 = \dfrac{BL+K}{1-BC}$ $\quad y_1 = \dfrac{CK+L}{1-BC}$



4.1) if $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3t-5 \\ -6t+4 \end{bmatrix}$, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$.

then
$$\begin{bmatrix} 3 \\ -6 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3t-5 \\ -6t+4 \end{bmatrix} + \begin{bmatrix} 9t \\ 0 \end{bmatrix}$$
$$\overset{?}{=} \begin{bmatrix} 3t-5+2(-6t+4)+9t \\ 2(3t-5)+1(-6t+4)+0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$
, so it is a solution.

4) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$; $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$

$x' = -x \qquad x = C_1 e^{-t} \qquad x(0) = 2 \Rightarrow C_1 = 2 \qquad x = 2e^{-t}$

$y' = y \qquad y = C_2 e^{t} \qquad y(0) = 0 \Rightarrow C_2 = 0$

then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}$

14) To see that this is a solution, use

$\begin{bmatrix} x' \\ y' \end{bmatrix} = c_1 \begin{bmatrix} -e^{-t}(2) \\ -e^{-t}(-3) \end{bmatrix} + c_2 \begin{bmatrix} 4e^{4t} \\ 4e^{4t} \end{bmatrix} + \begin{bmatrix} (-5)e^{t} \\ (3)e^{t} \end{bmatrix}$

and plug $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ into the system.

if $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, ~~then~~

$\begin{bmatrix} 0 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

and thus $c_1 = 2$, $c_2 = 1$.

15) $y'' + p(t)y' + q(t)y = r(t)$

rewrite as system

$v = y'$

$v' = -p(t)v - q(t)y + r(t)$ then $\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ r(t) \end{bmatrix}$

4.2) #3) $\begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}$

$$c(s) = s^2 - (-3)s + (-4)$$
$$= s^2 + 3s - 4$$
$$= (s+4)(s-1)$$

roots $s_1 = -4 \quad s_2 = 1$

$$\begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} = (-4)\begin{bmatrix} h \\ k \end{bmatrix}$$

$\begin{pmatrix} h+k \\ -4k \end{pmatrix} = \begin{pmatrix} -4h \\ -4k \end{pmatrix}$ let $h=1$ then $k=5$ so $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is a char. vector for $s_1 = -4$

$$\begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} = (1)\begin{matrix} h \\ k \end{matrix}$$

$\begin{bmatrix} h+k \\ -4k \end{bmatrix} = \begin{matrix} h \\ k \end{matrix}$ let $h=1$ then $k=0$ so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a char vect. for $s_2 = 1$

4) $\begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$ $c(s) = s^2$

$s_1 = 0$ is the char value.

so $\begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} = 0$ let $h=1$ then $k=-1$. $\underline{\underline{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}}$

8)

$$v' = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} v$$

for $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$, $c(s) = s^2 - 1 = (s+1)(s-1)$.

then let $s_1 = 1$ $\quad s_2 = -1$

Then we can find $b_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

so $v(t) = c_1 e^{(1)t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{(-1)t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

15) $v' = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} v \qquad v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then $c(s) = s^2 + 4s + 4 = (s+2)^2$

so $s = -2$ is a repeated characteristic value.

then let $\vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Clearly $A\vec{c} \neq (-2)\vec{c}$

so $\vec{b} = A\vec{c} - (-2)\vec{c} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

then the general solution is $v(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

using $v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we find $c_1 = -1$ and $c_2 = 1$

16) similar method: $s_1 = \frac{3+\sqrt{5}}{2} \quad s_2 = \frac{3-\sqrt{5}}{2}$

$\vec{b}_1 = \begin{bmatrix} \sqrt{5}-1 \\ 2 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix}$

$v(t) = c_1 e^{s_1 t} \vec{b}_1 + c_2 e^{s_2 t} \vec{b}_2$ and using the initial condition,

$c_1 = \frac{\sqrt{5}+1}{2} \quad c_2 = 1.$