1.3.1. *Find the general solution of the ODE $y' = 3t - 4y$ on the interval $(-\infty, \infty)$. Then find the particular solution that satisfies the initial condition $y(0) = 0$.*

Rewriting as $y' + 4y = 3t$, we multiply by the factor $e^{\int 4dt} = e^4 t$ to get $e^{4t}y' + 4e^{4t}y = 3te^{4t}$. This integrating factor should allow simplification of the left side of the equation, and indeed it becomes $\frac{d}{dt}e^{4t}y$. Integrating both sides with respect to t yields that $e^{4t}y = \int 3te^{4t}dt$. Integration by parts of the right side yields $\frac{3}{4}te^{4t} - \frac{3}{16}e^{4t} + C$, so $y = \frac{3t}{4} - \frac{3}{16} + Ce^{-4t}$, which is valid over the entire interval. Plugging in $t = 0$ and $y = 0$ yields $0 = 0 - \frac{3}{16} + C$, so $C = \frac{3}{16}$ and the solution to the initial value problem is $y = \frac{3t}{4} + \frac{3}{16}(e^{-4t} - 1)$.

1.3.5. *Find the general solution of the ODE $y' + 2ty = e^{-t^2}$ on the interval $(-\infty, \infty)$. Then find the particular solution that satisfies the initial condition $y(0) = 0$.*

We multiply by the factor $e^{\int 2tdt} = e^{t^2}$ to get that $e^{t^2}y' + 2te^{t^2}y = 1$. The left side of this equation simplifies to $\frac{d}{dt}e^{t^2}y$, so integrating both sides with respect to t gives $e^{t^2}y = t + C$, so $y = te^{-t^2} + Ce^{-t^2}$, which is valid over the entire interval. Plugging in $t = 0$ and $y = 0$ yields $0 = 0 + C$, so $C = 0$ and the solution to the initial value problem is $y = te^{-t^2}$.

1.3.13. *A penny is heated to $800°\,C$ and is then allowed to cool. The temperature after a minute is $600°\,C$, and the room temperature is $20°\,C$. When will it be safe to pocket the coin (the temperature should be less than $50°\,C$)?*

By Newton's law of cooling, the temperature y is governed by the diffferential equation $y' = -k(y - 20)$ for some constant k. Rearranging this yields $y' + ky = 20k$, and multiplication by the factor $e^{\int kdt} = e^{kt}$ gives us $\frac{d}{dt}e^{kt}y = 20ke^{kt}$, which upon integration with respect to t gives $e^{kt}y = \int 20ke^{kt}dt = 20e^{kt} + C$. Thus $y = 20 + Ce^{-kt}$, and it remains only to determine the values of C and k to completely determine the behavior of y. We know that $y(0) = 800$ and $y(1) = 600$, so $800 = 20 + C$ and $600 = 20 + Ce^{-k}$; the former gives us $C = 780$ and the latter $k = -\ln\frac{580}{780} \approx 0.296$. Now to solve the question actually asked, we must find a t such that $y(t) = 50$, that is, $50 = 20 + 780e^{-0.296t}$, which when solved gives $t = \frac{-1}{0.296}\ln\frac{30}{780} \approx 11$, so after 11 minutes it would be safe to touch the penny.

1.3.24. *Find the periodic solution of each of the following ODEs and show that it is stable.*

(c) $y' + 5y = \cos(\pi t)$.
We multiply by the integrating factor $e^{\int 5dt} = e^{5t}$ to get $e^{5t}y' + 5e^{5t}y = e^{5t}\cos(\pi t)$. The left side of this is equal to $\frac{d}{dt}e^{5t}y$, so integrating both sides with respect to t gives $e^{5t}y = \int e^{5t}\cos(\pi t) = \frac{e^{5t}}{\pi^2+25}(5\cos\pi t + \pi\sin\pi t) + C$, as determined by a computer-algebra system (alternatively, a table of integrals or integration by parts can be used). Then $y = \frac{5\cos\pi t + \pi\sin\pi t}{\pi^2+25} + Ce^{-5t}$, so over the long term y is approximately equal to $\frac{5\cos\pi t + \pi\sin\pi t}{\pi^2+25}$ regardless of starting value, since the exponential term approaches zero as t grows.

(f) $y' + py = \cos(\omega t)$, *where $p > 0$ is a constant. What would happen if $p < 0$?*

We multiply by the integrating factor $e^{\int p dt} = e^{pt}$ to get $e^{pt}y' + pe^{5t}y = e^{pt}\cos(\omega t)$. The left side of this is equal to $\frac{d}{dt}e^{pt}y$, so integrating both sides with respect to t gives $e^{pt}y = \int e^{pt}\cos(\omega t) = \frac{e^{pt}}{\omega^2+p^2}(p\cos\omega t + \omega\sin\omega t) + C$, as determined by a computer-algebra system (alternatively, a table of integrals or integration by parts can be used). Then $y = \frac{p\cos\omega t+\omega\sin\omega t}{\omega^2+p^2} + Ce^{-pt}$, so over the long term y is approximately equal to $\frac{p\cos\omega t+\omega\sin\omega t}{\omega^2+p^2}$ regardless of starting value, since the exponential term approaches zero as t grows. This would not be the case for $p < 0$, however, in which the exponential term is either constant (in the case $p = 0$) or grows without bound; In the latter case the periodic solution is unstable, since solutions tend to vary from it by increasing amounts as t grows, and in the $t = 0$ case there are an infinitude of periodic solutions, which are neither stable nor unstable.

2.1.5. *An object in free fall is subject to gravitational force of mg and frictional force of $-bv$. Show that the terminal velocity v_∞ satisfies the equation $mg - bv_\infty = 0$, and therefore $v_\infty = \frac{mg}{b}$.*

The differential equation expressing the velocity of the object is $v' = mg - bv$. The limiting velocity is a velocity at which $v' = 0$, since such a condition mandates a stable value which under normal conditions solution curves will not cross. Thus the terminal velocity is given by $0 = mg - bv_\infty$.

2.1.8. *Find a family of solutions for the ODE $y' = 3t^2y^2$. Then find the particular solution that satisfies the initial condition $y(0) = -1$ and specify its domain.*

We rearrage this equation into the separated form $\frac{dy}{y^2} = 3t^2dt$, which upon evaluation of the implicit integrals yields $\frac{-1}{y} = t^3 + C$, so $y = \frac{-1}{t^3+C}$ is a solution on either of the domains $(-\infty, -\sqrt[3]{C})$ or $(-\sqrt[3]{C}, \infty)$ (since the solution is undefined when $t^3 + C = 0$). Given the initial condition $y(0) = -1$, we evaluate the general form to obtain $-1 = \frac{-1}{C}$ so that $C = 1$, so our equation is $y = \frac{-1}{t^3+1}$ on either the domain $(-\infty, -1)$ or $(-1, \infty)$. Since we were given a value of y for $t = 0$, we must be considering the latter domain.

2.1.12. *Find a family of solutions for the ODE $dy = (y^2 + 1)dt$. Then find the particular solution that satisfies the initial condition $y(0) = 1$ and specify its domain.*

We rearrage this equation into the separated form $\frac{dy}{y^2+1} = dt$, which upon evaluation of the implicit integrals yields $\tan^{-1} y = t + C$, so $y = \tan(t + C)$ is a solution on any domain where y is defined; in particular, the boundaries of the domain are the points where $t + C = \pi n + \frac{\pi}{2}$. Given the initial condition $y(0) = 1$, we evaluate the general form to obtain $1 = \tan(0 + C)$ so that $C = \frac{\pi}{4}$, so our equation is $y = \tan(t + \frac{\pi}{4})$ on some domain which does not include values $t = \pi n + \frac{\pi}{4}$. In particular, to include $t = 0$, which is our initial condition, our maximal domain is $\left(\frac{-3\pi}{4}, \frac{\pi}{4}\right)$.

2.1.16. *If an object is initially moving upward (we consider upward velocity to be negative), then the force of air resistance will be downward (positive), and the ODE*

$$v' = g + \frac{k}{m}v^2$$

will apply as long as $v \leq 0$, but when $v > 0$, the sign of the quadratic term should be changed to minus.

(a) *Show that the ODE*

$$v' = g - \frac{k}{m} v|v|$$

is applicable to both upward and downward motion.

We note that $v|v| = v^2$ when $v > 0$ and $v|v| = -v^2$ when $v < 0$ (and all three are of course identical for $v = 0$), so this ODE satisfies the description given.

(b) *An object is thrown upward with an initial velocity of 100 meters/second ($v(0) = -100$) subject to the differential equation*

$$v' = 9.8 + 9.8 \times 10^{-4} v^2.$$

Give its velocity as a function of time for the duration of its upward motion.

We rearrage this equation into the separated form $\frac{dv}{(9.8\times 10^{-4})(100^2+v^2)} = dt$, so integrating both sides and rearranging gives $\tan^{-1} \frac{v}{100} = 9.8 \times 10^{-2} t + C$. Plugging in the values $t = 0$ and $v = -100$ yields $\tan^{-1}(-1) = C$, so $C = \frac{-\pi}{4}$, and the equation becomes

$$v = 100 \tan\left(9.8 \times 10^{-2} t - \frac{\pi}{4}\right).$$

(c) *Adapt the solution of Example 2.1.4 to obtain the velocity as a function of time for downward motion.*

From the previous part of this exercise, we know that the velocity is zero when $9.8 \times 10^{-2} t - \frac{\pi}{4} = 0$, so we reach zero velocity at time $t = \frac{\pi}{39.2\times 10^{-2}}$. Example 2.1.4 gives an equation for an object's fall from rest starting at time $t = 0$, so all that need be done is to "recenter" the equation given so that $t = \frac{\pi}{39.2\times 10^{-2}}$ corresponds to $t = 0$ in the original example:

$$v = 100 \frac{e^{0.0196(t - \frac{\pi}{39.2\times 10^{-2}})} - 1}{e^{0.0196(t - \frac{\pi}{39.2\times 10^{-2}})} + 1}.$$

2.3.18. (a) *Show that the solution of the IVP*

$$y' = f(x); y(a) = b,$$

is

$$y(x) = b + \int_a^x f(u)du.$$

We rearrage this equation into the separated form $dy = f(x)dx$, so $y = \int_0^x f(u)du + C$. Our lower bound here is arbitrary, since any change in it can be incorporated into our added constant C, so we might as well make $y = \int_a^x f(u)du + C$. Applying the initial condition $y(a) = b$, we find that $b = \int_a^a f(u)du + C = C$. Thus our solution is $y(x) = b + \int_a^x f(u)du$.

(b) *Show that approximating the solution to the aforementioned IVP by Euler's method amounts to calculating a Riemann sum for the integral in question.*

Euler's method gives us that $y(a+kh) \approx y\big(a+(k-1)h\big) + hy'\big(a+(k-1)h\big) = y\big(a+(k-1)h\big) + hf\big(a+(k-1)h\big)$. Arguing inductively will show that, if we derive each $y(a+kh)$ in this manner, then $y(a+kh) \approx y(a) + \sum_{n=1}^{k} hf\big(a+(n-1)h\big)$. The former addend is simply b, and the latter is clearly the left-hand Riemann sum with step h of the integral $\int_a^{a+kh} f(u)du$.

2.3.21. *Let $y = \phi(t)$ denote the solution of the IVP,*

$$y' = t + y; y(0) = 0.$$

Since the differential equation is linear, you can solve this IVP and find a formula for $\phi(t)$. This problem asks you to calculate $\phi(1)$ and to approximate $\phi(1)$ by using Euler's method. The purpose is to see how fast the approximation improves when the time step is reduced.

Use Euler's method with time steps $h = 1$, 0.5, 0.25, 0.1, 0.05, 0.02, and 0.01 to approximate $\phi(1)$, and let $E(h) = |\phi(1) - y_N|$ (where $Nh = 1$) denote the approximation error obtained with time step h. Plot a graph of $E(h)$ as a function of h.

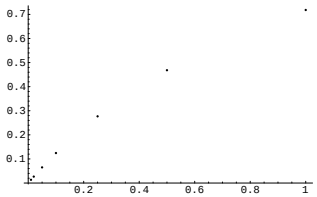
The solution of this differential equation follows a familiar pattern: we rearrange the differential equation as $y' - y = t$ and multiply by the integrating factor $e^{\int -1dt} = e^{-t}$ to yield $e^{-t}y' - e^{-t}y = e^{-t}t$. The left side is simply $\frac{d}{dt}e^{-t}y$, so integrating both sides with respect to t results in the equation $e^{-t}y = C - te^{-t} - e^{-t}$, or $y = Ce^t - t - 1$. Plugging in the initial conditions will yield that $C = 1$, so $\phi(t) = e^t - t - 1$, and in particular $\phi(1) = e - 2 \approx 0.71828$.

Below a table indicates the steps in estimating $y(1)$ using Euler's method with step $h = 0.25$. Similar procedures are followed for other values of h, and may be determined by hand or (in the case of very small h) with a computer or calculator.

t	$y(t)$	$y'(t) = y + t$	$\Delta y = hy'(t)$
0	0	0	0
0.25	0	0.25	0.0625
0.50	0.0625	0.5625	0.140625
0.75	0.203124	0.953125	0.23828125
1.00	0.44140625		

The calculation being used to redetermine y at each line is $y(x+h) = y(x) + \Delta y(x)$. Applying this for several values of h we get the following estimations of $\phi(1)$:

h	N	y_N	$E(h)$
1	1	0	0.718281
0.50	2	0.25	0.468282
0.25	4	0.441406	0.276878
0.1	10	0.593742	0.124539
0.05	20	0.653298	0.064984
0.02	50	0.691588	0.026693
0.01	100	0.704814	0.013468



2.3.25. *The solution $y = e^{-100t}$ of the initial value problem*

$$y' = -100y; y(0) = 1$$

converges to 0 very rapidly. This is apparent even if we restrict our attention to $0 \le t \le 1$, since $e^{-100\times 1}$ is of the order of magnitude of 10^{-43}.

(a) *Use Euler's method with $h = 0.1$ to approximate the solution of the initial value problem for $0 \le t \le 1$. Does your computed solution appear to converge to 0?*

Using Euler's method, the apparent value of y diverges rapidly and alternates sign; $y(0.1)$ is computed to be -9, $y(0.2)$ is computed to be 81, and so forth, until $y(1)$ is computed to be approximately 3.5×10^9.

(b) *How small should h be to insure that the solution converges to 0?*

To converge to zero, we want the absolute values of the terms to diminish. Note that Euler's method assumes that $y(t+h) = y(t) + hy'(t)$, which in this particular case is $y(t) - 100hy(t) = (1 - 100h)y(t)$. Thus the absolute value of y is diminishing if $|1 - 100h| < 1$, so $1 - 100h$ must be greater than -1, giving that h must be greater than $\frac{1}{50} = 0.02$.

2.3.26. *One approach that avoids erratic behavior observed in the solution computed in Exercise 2.3.25 is to use* backward *differences instead of forward differences with Euler's method. Thus we use the difference equation*

$$\nabla y_{m+1} = hf(t_{m+1}, y_{m+1}) \text{ for } m \ge 0,$$

where $\nabla y_{m+1} = y_{m+1} - y_m$, as a model for the differential equation $y' = f(t, y)$. This difference scheme is said to be implicit *because it defines y_{m+1} implicitly by means of the relation*

$$y_{m+1} - y_m - hf(t_{m+1}, y_{m+1}) = 0.$$

Test the backward Euler method by approximating the solution of the initial value problem in Exercise 2.3.25, again with $h = 0.1$, for $0 \le t \le 1$.

First we must solve the implicit relation given: $y_{m+1} - y_m - h(-100y_{m+1}) = 0$, which will give that $(1 + 100h)y_{m+1} = y_m$, so $y_{m+1} = \frac{y_m}{1+100h}$, which inductively extends to tell us that $y_m = \frac{y_0}{(1+100h)^m}$, so in the case $h = 0.1$, finding $y(1) = y_{10}$ gives us a very small number indeed (approximately 3.9×10^{-11}).

2.4.2. *Solve the IVP $y' - 5y = 25t$; $y(0) = 10$ and find the largest interval on which the solution is defined.*

By the existence theorem, this differential equation's solution should be valid over all values of t. We multiply by the factor $e^{\int -5dt} = e^{-5t}$ to get $e^{-5t}y' - 5e^{-5t}y = 25e^{-5t}t$; the left side simplifies to $\frac{d}{dt}e^{-5t}y$, so integration of both sides with respect to t gives $e^{-5t}y = -5te^{-5t} - e^{-5t} + C$ (with the value on the right obtained via integration by parts). The general solution is thus $y = -5t - 1 + Ce^{5t}$, and evaluating at the initial value $y(0) = 10$ gives $10 = -1 + C$, so $C = 11$ and the solution, valid at all values of t, is $y = 9e^{5t} - 5t - 1$.

2.4.22. *Determine which of the following functions satisfy a Lipschitz condition with respect to y on the domain*

$$\mathcal{D} = \{(t, y) : -1 < t < 1, -1 < y < 1\},$$

and find a Lipschitz constant for those that do.

(a) $f(t, y) = t - y^2$.
Note that $|(t - y_2^2) - (t - y_1^2)| = |y_1^2 - y_2^2| = |y_1 - y_2| \cdot |y_1 + y_2|$, so our Lipschitz constant (if such exists) would be given by the maximum value of $|y_1 + y_2|$, which is 2. Since we have just shown that $|f(t, y_2) - f(t, y_1)| \leq 2|y_2 - y_1|$, the Lipschitz condition is satisfied.

(b) $f(t, y) = 4ty$.
Note that $|4ty_2 - 4ty_1| = 4|t| \cdot |y_2 - y_1|$, so our Lipschitz constant (if such exists) would be given by the maximum value of $4|t|$, which is 4. Since we have just shown that $|f(t, y_2) - f(t, y_1)| \leq 4|y_2 - y_1|$, the Lipschitz condition is satisfied.

(c) $f(t, y) = |y|$.
When y_1 and y_2 have the same sign, then $\big||y_1| - |y_2|\big| = |y_1 - y_2|$, and when y_1 and y_2 have opposite signs, $\big||y_1| - |y_2|\big| = |y_1 + y_2| \leq |y_1 - y_2|$, so in each case $|f(t, y_2) - f(t, y_1)| \leq |y_2 - y_1|$, so the Lipschitz condition is satisfied with constant 1.

(d) $f(t, y) = \lfloor y \rfloor$.
Note that $\lfloor 1 \rfloor = 1$, and that $\lfloor 1 - \epsilon \rfloor = 0$ for small positive ϵ. Thus, taking $y_2 = 1$ and $y_1 = 1 - \epsilon$, we find that $|f(t, y_2) - f(t, y_1)| = 1$, while $|y_2 - y_1| = \epsilon$, so it is necessary, if the Lipschitz condition is satisfied, that $K > \frac{1}{\epsilon}$. However, since ϵ can be as smal as we like, tihs would force K to be arbitrarily large, so the Lipschitz condition is unsatisfied.

(e) $f(t, y) = \lfloor t \rfloor$.
Since $f(t, y)$ does not depend on y, it follows that $f(t, y_2) - f(t, y_1) = 0$, so $|f(t, y_2) - f(t, y_1)| \leq 0|y_2 - y_1|$, satisfying the Lipschitz condition with constant 0.