HW #3

7) $f(x) = e^{5x}$ $\qquad f(0)=1$

$f'(x) = 5e^{5x}$ $\qquad f'(0) = 5$

$f''(x) = 5^2 e^{5x}$ $\qquad f''(0) = 5^2$

$f^{(3)}(x) = 5^3 e^{5x}$ $\qquad f'''(0) = 5^3$

$$e^{5x} = \frac{1 \cdot x^0}{0!} + \frac{5 \cdot x^1}{1!} + \frac{5^2 x^2}{2!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{5^n x^n}{n!}$$

by the ratio test, $\lim_{n\to\infty} \left| \frac{\frac{5^{n+1} x^{n+1}}{(n+1)!}}{\frac{5^n x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{5x}{n+1} \right| = 0 < 1.$

so this series is convergent for all x.

19) $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

note that for $f(x) = \cos x$, $|f^{n+1}(x)| < 1$ for any n and any x.

then by the Taylor Inequality,

$$|R_n(x)| \le \frac{1 \cdot |x|^{n+1}}{(n+1)!} \text{ for all n and all x.}$$

but then $\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$

Thus $\lim_{n\to\infty} |R_n(x)| = 0$ so $f(x)$ is equal to its Taylor series.

4a) Integrate as a power series:

$$\int \frac{\sin x}{x}\,dx$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \frac{\sin x}{x} = \frac{1}{x}\cdot\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

Then $\displaystyle\int \frac{\sin x}{x}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+1)!}$

4b) $\displaystyle\int \frac{e^x-1}{x}\,dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\frac{e^x-1}{x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Thus $\displaystyle\int \frac{e^x-1}{x}\,dx = \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\,dx = \sum_{n=1}^{\infty} \frac{x^n}{n\cdot n!}$

5a). As $\displaystyle e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$

$$e^3 = 1 + \frac{3}{1} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots$$

so $\displaystyle e^3 - 1 = \frac{3}{1} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots$

2) a) $f(x) = 1/x$ $\quad f(1) = 1$

$f'(x) = \frac{-1}{x^2}$ $\quad f'(1) = -1$

$f''(x) = \frac{2}{x^3}$ $\quad f''(1) = 2$

$f'''(x) = \frac{-6}{x^4}$ $\quad f'''(1) = -6$

$T_0(x) = 1$

$T_1(x) = 1 - (x-1)$

$T_2(x) = 1 - (x-1) + \frac{2(x-1)^2}{2!} = 1-(x-1) + (x-1)^2$

$T_3(x) = 1 - (x-1) + (x-1)^2 - \frac{6(x-1)^3}{3!}$

$= 1 - (x-1) + (x-1)^2 - (x-1)^3$

b, c)

27) $\sin x \approx x - \frac{x^3}{6}$ with $|\text{error}| < .01$

By the Alternating Series Estimation Thm,

If $S = \sum_{k=0}^{\infty} (-1)^k b_k$ with $\lim_{k\to\infty} b_k = 0$ and $b_k \geq b_{k+1}$,

for ~~[illegible]~~ $S_n = \sum_{k=0}^{n} (-1)^k b_k$, $|S - S_n| \leq b_{n+1}$.

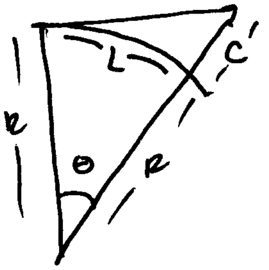
Thus $\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq \frac{x^5}{5!}$

To find the correct values of x,

let $\left|\frac{x^5}{5!}\right| < \frac{1}{100} \Rightarrow |x| < \left(\frac{120}{100}\right)^{1/5}$

$\Rightarrow |x| \leq 1.037$

35)



by basic trig, $L = R\theta$. so $\theta = L/R$. Now $\sec\theta = \frac{R+C}{R}$ in this triangle, so $R\sec\theta = R+C$, and thus

$$C = R\sec(L/R) - R$$

b) The Taylor polynomial for $\sec x$ of degree 4 about 0 is given by

$$T_4(x) = 1 + \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4$$

so $\sec\theta \approx 1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4$

Plugging into our expression from a),

$$C \approx R\left(1 + \tfrac{1}{3}(L/R)^2 + \tfrac{5}{24}(L/R)^4\right) - R$$

$$= \frac{L^2}{2R} + \frac{5L^4}{24R^3}$$

c) exact): .78500996544 km

$T_4(x)$: .78500995736 km.

9) $y^3 + (y')^3 = 1.$

1st order ODE

12) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

2nd order PDE

15) $y = C\sin(t) + D\cos(t)$

$y' = C\cos(t) - D\sin(t)$

$y'' = -C\sin(t) - D\cos(t)$

Clearly $y + y'' = 0.$

a) $y(0) = 0$

So $0 = y(0) = C\sin(0) + D\cos(0) = D.$

Then $D = 0$ and $y(t) = C\sin(t)$

b) $y(0) = 0 \quad y(\pi) = 0$

from a), $y(0) = 0 \Rightarrow y = C\sin t.$

then $0 = y(\pi) = C\sin(\pi) = 0$, which gives no new information.

So $y = C\sin t.$

c) $y(0) = 0 \quad y(\pi/6) = 1$

from a) $y = C\sin t.$ Now $1 = y(\pi/6) = C\sin \pi/6 = C/2.$

Then $C = 2$ and $y = 2\sin t.$

d) $y(0)=1 \quad y'(0)=-1.$

~~[illegible]~~

$$1 = y(0) = C\sin(0) + D\cos(0) = D$$

so $D=1 \quad y = C\sin t + \cos t$

$$-1 = y'(0) = C\cos(0) - \cancel{D}\sin(0)$$
$$= C$$

so $C = -1.$

Then $y = -\sin t + \cos t$

17) $y = Ce^{t^2}$

$$y' = (2t)Ce^{t^2}$$

$$y' = (2t)\,Ce^{t^2} = (2t)\,(Ce^{t^2}) = (2t)y$$

So $y = Ce^{t^2}$ solves the ODE.

$3 = y(0) = Ce^{(0)^2} = C$ So $C=3$ and

$y = 3e^{t^2}$ solves the IVP.

8) a) $y' - 2y = 0 \quad y(0) = 1$

Try $y = Ce^{k(t)}$

$k'(t)Ce^{k(t)} - 2Ce^{k(t)} = 0$

so $k'(t) = 2 \Rightarrow k(t) = 2t$

then $y(t) = Ce^{2t}$

$1 = y(0) = Ce^{0} \Rightarrow C = 1$

then $y = e^{2t}$ solves the IVP.

b) try $y = Ce^{k(t)} \quad y(0) = 1$

$y' + 2y = 0 \qquad k'(t)Ce^{k(t)} + 2Ce^{k(t)} = 0$

then $k'(t) = -2 \quad k(t) = -2t$

$y = Ce^{-2(t)}$

$1 = y(0) = Ce^{-2(0)} = C$

so $C = 1$ and $y = e^{-2t}$.

12) $y' + ty = 0$ Try $y = Ce^{k(t)} \quad y(1) = 1$

$k'(t)Ce^{k(t)} + tCe^{k(t)} = 0 \Rightarrow k'(t) = -t$ so $k(t) = -t^2/2$

then $y = Ce^{-t^2/2} \quad 1 = y(1) = Ce^{-1/2} \Rightarrow C = e^{1/2}$

so $y = e^{1/2}e^{-t^2/2} = e^{-t^2/2 + 1/2}$

16) tree dies at $t=0$.

then $y(0)=1$.

Now exp. decay is given by $y = Ce^{kt}$ for some negative k.

$1 = y(0) = Ce^{k(0)} = C$ so $C=1$.

$y(t) = e^{kt}$

The half life of C^{14} is 5730 years, so

$y(5730) = 1/2$.

$1/2 = y(5730) = e^{k(5730)}$

So $\ln(1/2) = k(5730)$

$$k = \frac{\ln(1/2)}{5730}$$

which gives $y = e^{-.000121t}$ and $y(0)=1$. ~~model equation.~~ ~~Th... solve the~~

$$y' = (-.000121)e^{-.000121t} = (-.000121)y.$$

17) $1922-(-1325) = 3247$

so $y(0)=1$, we need $y(3247)$. from #16,

$y = e^{-.000121t}$ so $y(3247) = e^{-.000121(3247)} = .675$