11.1.1. (a) *What is a sequence?*

A sequence is a list of numbers with an order. An infinite sequence is an infitine list of numbers with an order.

(b) *What does it mean to say that* $\lim_{n\to\infty} a_n = 8$*?*

This means, informally speaking, that the terms of the sequence a_n get closer and closer to 8. Formally, it means that for any positve small ϵ, we can find an N such that the Nth and subsequent terms of the sequence are within ϵ of 8.

(c) *What does it mean to say that* $\lim_{n\to\infty} a_n = \infty$*?*

This means, informally speaking, that the terms of the sequence a_n get larger and larger without bound. Formally, it means that for any arbitrarily large E, we can find an N such that the Nth and subsequent terms of the sequence are greater than E.

11.1.12 *Find a formula for the general term* a_n *of the sequence* $\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}\}$*, assuming that the pattern of the first few terms continues.*

The sequence suggested alternates between positive and negative, with numerators increasing linearly and denominators being squares, which suggests $a_n = \frac{(-1)^n n}{(n+1)^2}$.

11.1.19 *Determine whether the sequence* $a_n = \frac{2^n}{3^{n+1}}$ *converges or diverges. If it converges, find the limit.*

We may rewrite this as $a_n = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n$, which is a geometric sequence with ratio $\frac{2}{3}$, which thus converges to zero, since the absolute value of the ration is less than 1.

11.1.22 *Determine whether the sequence* $a_n = \frac{(-1)^n n^3}{n^3+2n^2+1}$ *converges or diverges. If it converges, find the limit.*

By Theorem 3 we know that $\lim_{n\to\infty} |a_n| = \lim_{x\to\infty} \frac{x^3}{x^3+2x^2+1} = 1$, so the limit of the absolute value of this sequence is 1. We know since one term alternates signs and the others are positive, that the terms of this sequence alternate signs. Thus, since the absolute value of this sequence is approaching 1, the terms of this sequence are alternating between positive and negative values approaching 1 and -1. Since the sequence is not approaching a single value, it diverges.

11.1.56 *Determine whether the sequence* $a_n = \frac{2n-3}{3n+4}$ *is increasing, decreasing, or not monotonic. Is the sequnce bounded?*

To determine whether this increases or decreases, we must find the relative magnitude of $a_n = \frac{2n-3}{3n+4}$ and $a_{n+1} = \frac{2(n+1)-3}{3(n+1)+4} = \frac{2n-1}{3n+7}$. Putting these in terms of common denominators, these are $\frac{6n^2+5n-21}{9n^2+33n+28}$ and $\frac{6n^2+5n-4}{9n^2+33n+28}$ respectively; the latter is clearly larger, so $a_n < a_{n+1}$ for arbitrary n. This is thus an increasing sequence. It is bounded below by a_1 by virtue of being increasing, and it is bounded above by $\frac{2}{3}$ since $\frac{2n-3}{3n+4} < \frac{2n}{3n}$.

11.1.62 *A sequence* $\{a_n\}$ *is given by* $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

(a) *By induction or otherwise, show that* $\{a_n\}$ *is increasing and bounded above by 3. Apply Theorem 11 to show that* $\lim_{n\to\infty} a_n$ *exists.*

Let us inductively assume $0 < a_n < 2$ (which we know is true for the case $n = 1$); then by adding 2 and taking the square root, we find that $\sqrt{2} < \sqrt{2 + a_n} < 2$; thus it is the case that $0 < a_{n+1} < 2$, since $0 < \sqrt{2}$. Thus the sequence is bounded between 0 and 2 (and thus bounded above by 3 as well). To show that it is increasing, let us consider the inductive assumption $a_n < a_{n+1}$, which is clearly true for $n = 1$; then let us derive the next case by adding 2 and taking the square root of each side, so that $\sqrt{2 + a_n} < \sqrt{2 + a_{n+1}}$, which implies that $a_{n+1} < a_{n+2}$. Thus this sequence is increasing. Since it is increasing and bounded, it follows that its limit exists.

(b) *Find* $\lim_{n\to\infty} a_n$.

Since we know that the limit actually exists, we may let $L = \lim_{n\to\infty} a_n$. Then it also is clear that

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2 + L}.$$

Thus the limit satisfies $L = \sqrt{2 + L}$. Squaring both sides of this expression and rearranging yields the quadratic $L^2 - L - 2 = 0$, which has solutions $L = 2$ and $L = -1$. $L = -1$ is a spuriouls root introduced by squaring; the only solution of the original equation is 2, so $L = 2$.

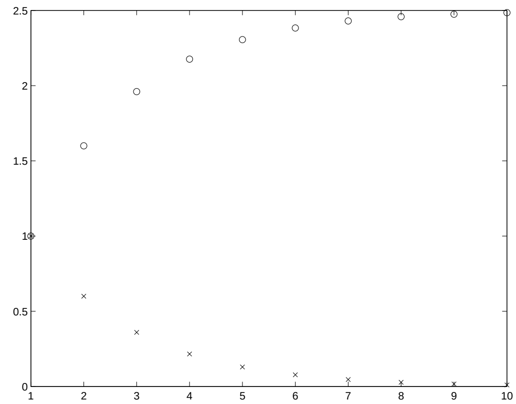
11.1.68 *Use the* $\epsilon - N$ *definition of a limit directly to prove that* $\lim_{n\to\infty} r^n = 0$ *when* $|r| < 1$.

The definition of the limit in question is that, for any small ϵ, there is an N such that for $n > N$, $|r^n - 0| < \epsilon$; we shall attempt to find a plausible value of N, then show that it satisfies these requirements. The expression above can be rearranged into $|r|^n < \epsilon$; taking the logarithm of each side (to isolate the n), we find that $n \ln |r| < \epsilon$. Note that since $|r| < 1$, $\ln |r|$ is negative, so division by it reverses the inequality: $n > \frac{\epsilon}{\ln |r|}$. It thus appears to be sensible to let $N = \frac{\epsilon}{\ln |r|}$, and indeed we can show that this value of N satisfies the condition given: $|r^n| < |r^N| = |r^{\frac{\epsilon}{\ln |r|}}| = |\epsilon|$.

11.2.2 *Explain what it means to say that* $\sum_{n=1}^{\infty} a_n = 5$.

This means that the sequence of partial sums $\{\sum_{i=1}^{n} a_i\}$ converges to 5.

11.2.6 *Find at least ten partial sums of the series* $\sum_{n=1}^{\infty} (0.6)^{n-1}$. *Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.*



This sequence appears to converge, since the partial sums appear to level off (although this appearance can deceive). We know it converges, however, since it is a geometric series, and it converges to $\frac{1}{1-0.6} = \frac{1}{0.4} = 2.5$.

11.2.19 *Determine whether the series $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$ is convergent or divergent. If it is convergent, find its sum.*

The terms of this series can be written in the form $\frac{1}{3} \cdot \left(\frac{\pi}{3}\right)^n$, so this is a geometric series with ration $\frac{\pi}{3}$, which is greater than 1, and thus does not converge.

11.2.26 *Determine whether the series $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ is convergent or divergent. If it is convergent, find its sum.*

Let us break a term of this series into partial fractions: $\frac{2}{n^2+4n+3} = \frac{A}{n+1} + \frac{B}{n+3}$. Rearranging this, we find that $(A+B)n + 3A + B = 2$, which has the solution $A = 1$, $B = -1$, so $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$. Thus, if we expand several terms of the series written in theis form, we shall see that every term cancels except for the initial $\frac{1}{2}$ and $\frac{1}{3}$, and since the terms being cancelled are decreasing in magnitude, it will converge to the sum $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

11.2.37 *Express* $3.\overline{417} = 3.417417417417\ldots$ *as a ratio of integers.*

This series is $3 + \frac{417}{1000} + \frac{417}{1000000} + \cdots$, which we may rewrite as $3 + \sum_{n=1}^{\infty} 417 \left(\frac{1}{1000}\right)^n$. The sum here is a geometric series with ratio $\frac{1}{1000}$, which using the geometric sum formula we may evaluate to be $\frac{417 \cdot \frac{1}{1000}}{1 - \frac{1}{1000}} = \frac{417}{999}$. Thus the original expression evaluates to $3 + \frac{417}{999} = \frac{3414}{999}$.

11.2.59 *Prove that for $\sum_{n=1}^{\infty} a_n$ convergent, $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.*

Let $s_n = \sum_{i=1}^{n} a_i$. Then $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ by definition. It is also clear that $cs_n = c\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ca_i$, so $\lim_{n\to\infty} cs_n = \sum_{n=1}^{\infty} ca_n$. Thus:

$$c\sum_{n=1}^{\infty} a_n = c \lim_{n\to\infty} s_n = \lim_{n\to\infty} cs_n = \sum_{n=1}^{\infty} ca_n.$$

11.2.61 *If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent, show that the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is divergent (Argue by contradiction).*

Suppose to the contrary that $\sum_{n=1}^{\infty}(a_n+b_n)$ is convergent. Then, we know (via 11.2.59) that since $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty}(-1)a_n = \sum_{n=1}^{\infty} -a_n$. Then, since the sum of two convergent series converges, we may add together $\sum_{n=1}^{\infty}(a_n+b_n)$ and $\sum_{n=1}^{\infty} -a_n$ to get that $\sum_{n=1}^{\infty}(a_n+b_n-a_n) = \sum_{n=1}^{\infty} b_n$ converges, which contradicts our known fact that $\sum_{n=1}^{\infty} b_n$ diverges. Thus our premise that $\sum_{n=1}^{\infty}(a_n+b_n)$ is convergent must be false.

11.3.3 *Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent or divergent.*

By the integral test, this is convergent if and only if $\int_1^{\infty} \frac{1}{x^4}dx$ converges. Since this integral is $\left.\frac{1}{-3x^3}\right]_1^{\infty} = 0 - \frac{1}{-3}$, which converges, so the series converges as well.

11.3.22 *Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ is convergent or divergent.*

By the integral test, this is convergent if and only if $\int_1^{\infty} \frac{x}{x^4+1}dx$ converges. Using the substitution $u = x^2$, $du = 2xdx$, we may convert this integral to $\int_1^{\infty} \frac{1}{2(u^2+1)}du$, which evaluates to $\arctan u]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4}$, which converges.

11.3.28 *Find the values of p for which the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ is convergent.*

By the integral test, this is convergent if and only if $\int_1^{\infty} \frac{\ln x}{x^p}dx$ converges. We integrate by parts with the substitution $u = \ln x$, $dv = \frac{1}{x^p}dx$, which in all cases except $p = 1$ yields $du = \frac{1}{x}$ and $v = \frac{1}{-(p-1)x^{p-1}}$, so

$$\int_1^{\infty} \frac{\ln x}{x^p}dx = \left.\frac{\ln x}{-(p-1)x^{p-1}}\right]_1^{\infty} + \int_1^{\infty} \frac{1}{-(p-1)x^p}$$

The integral on the right side of this equation we know converges if and only if $p > 1$. The evaluation converges also as long as the term being evaluated approaches zero as x approaches infinity: this occurs as long as $p - 1 > 0$. Thus both terms converge only when $p > 1$ (and since both approach positive infinity when they diverge, there is no problem of uncertainty adding the divergent cases), so the convergence criterion appears to be that $p > 1$. However, we specifically removed the case $p = 1$ from our evaluation of the integral previously, so we must test it separately to determine convergence in this case. To integrate $\int_1^{\infty} \frac{\ln x}{x}dx$, we use the substitution $u = \ln x$, $du = \frac{1}{x}dx$ to convert this to $\int_1^{\infty} udu$, which clearly does not converge. Thus our surmise that only the $p > 1$ cases converge is correct.

11.3.32 *Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to three decimal places.*

We wish to find the partial sum s_n such that the remainder R_n is at most 0.001 (to be particularly cautious, we might want to use 0.0005; the choice of error margin which is regarded as "correct to three decimal places" varies depending on discipline). We know from integral-test methods that $R_n \leq \int_n^{\infty} \frac{1}{x^5}dx$, so it will suffice to find an n such that $\int_n^{\infty} \frac{1}{x^5}dx \leq 0.001$. Evaluating the integral on the left side yields $\frac{1}{4n^4} \leq 0.001$, which rearranged alegraically gives $250 \leq n^4$; since n must be an integer, we need n to be at least 4. Thus, a sufficient estimate of the sum is simply

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \approx 1.0363$$