6.3.1. *Calculate the Laplace transform $L(t^3 + 5t^2 + 2t - 1)$.*

Using linearity of the Laplace transform operator and the known transform $L(t^n) = \frac{n!}{s^{n+1}}$, we find that

$$L(t^3 + 5t^2 + 2t - 1) = L(t^3) + 5L(t^2) + 2L(t) - L(1) = \frac{6}{s^4} + \frac{10}{s^3} + \frac{2}{s^2} - \frac{1}{s}$$

6.3.13. *Calculate the Laplace transform of $f(t) = (te^t)^3$.*

Note that $f(t) = t^3e^{3t}$. From the table of Laplace transforms we know that $L(t^3) = \frac{3}{s^4}$ and $L(e^{3t}f(t)) = F(s-3)$, so $L(t^3e^{3t}) = \frac{3}{(s-3)^4}$.

6.3.31. *Use the Laplace transform method to solve the initial value problem $y'' + 4y' + 4y = 0$; $y(0) = 0$, $y'(0) = 1$.*

Taking the Laplace transform of this second order ODE yields the equation $s^2Y - sy(0) - y'(0) + 4(sY - y(0)) + 4Y = 0$. Substituting in the known values of $y(0)$ and $y'(0)$ yields $(s^2+4s+4)Y - 1 = 0$, so $Y = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$. Taking the inverse Laplace transform of this, $y = te^{-2t}$.

6.3.41. *Use the Laplace transform method to solve the initial value problem*

$$\begin{cases} x' = 3x - 2y; & x(0) = 3 \\ y' = 2x - y; & y(0) = 0 \end{cases}$$

Taking the Laplace transform of this system of equations yields the new system

$$\begin{cases} sX - 3 = 3X - 2Y \\ sY = 2X - Y \end{cases}$$

which we may rewrite in the more conventional form

$$\begin{cases} (s-3)X + 2Y = 3 \\ -2X + (s+1)Y = 0 \end{cases}$$

Multiplying the first equation by 2 and the second by $s-3$, then adding them together, yields the simpler equation $(4 + (s+1)(s-3))Y = 6$, so $Y = \frac{6}{s^2-2s+1} = \frac{6}{(s-1)^2}$. Substituting this into the second equation, $X = \frac{s+1}{2}Y = \frac{3s+3}{(s-1)^2}$. This is not in a form of which we can take the inverse Laplace transform, so we must utilize the partial fraction decomposition $\frac{3s+3}{(s-1)^2} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2}$, so $3s + 3 = A(s-1) + B$, so matching linear and constant terms, $A = 3$ and $-A + B = 3$, so the decomposition is $X = \frac{3}{s-1} + \frac{6}{(s-1)^2}$. We may now take the inverse Laplace transforms to find that $y = 6te^t$ and $x = 3e^t + 6te^t$.

6.3.44. *Let $f(t)$ be a function that has a Laplace transform, and let $F(s) = L(f)$. If k denotes a positive constant, prove that*

$$L[f(kt)] = \frac{1}{k}F\left(\frac{s}{k}\right).$$

Note that $L[f(kt)] = \int_0^\infty f(kt)e^{-st}dt$. Let $u = kt$, so $dt = \frac{1}{k}du$; substituting this into the above integral, we get $L[f(kt)] = \frac{1}{k}\int_0^\infty f(u)e^{-\frac{s}{k}u}du$. Note that $F(s) = \int_0^\infty f(u)e^{-su}du$, so by replacing s with $\frac{s}{k}$, we get the integrand in the previous expression. Thus, $L[f(kt)] = \frac{1}{k}F\left(\frac{s}{k}\right)$.

Use this result to derive formulas for the Laplace transforms of the following expressions.

(a) e^{kt}, *given that* $L(e^t) = \frac{1}{s-1}$.
Let $f(t) = e^t$, so $F(s) = \frac{1}{s-1}$. Then

$$L(e^{kt}) = L(f(kt)) = \frac{1}{k}F\left(\frac{s}{k}\right) = \frac{1}{k}\cdot\frac{1}{\frac{s}{k}-1} = \frac{1}{s-k}.$$

(b) $\cos(kt)$, *given that* $L[\cos(t)] = \frac{s}{s^2+1}$.
Let $f(t) = \cos(kt)$, so $F(s) = \frac{s}{s^2+1}$. Then

$$L[\cos(kt)] = L(f(kt)) = \frac{1}{k}F\left(\frac{s}{k}\right) = \frac{1}{k}\cdot\frac{\frac{s}{k}}{\frac{s^2}{k^2}+1} = \frac{s}{s^2+k^2}.$$

(c) $\sin(kt)$, *given that* $L[\cos(t)] = \frac{1}{s^2+1}$.
Let $f(t) = \sin(kt)$, so $F(s) = \frac{1}{s^2+1}$. Then

$$L[\sin(kt)] = L(f(kt)) = \frac{1}{k}F\left(\frac{s}{k}\right) = \frac{1}{k}\cdot\frac{1}{\frac{s^2}{k^2}+1} = \frac{k}{s^2+k^2}.$$

6.4.1. *Find the inverse Laplace transform of the expression* $\frac{6}{s^3+s^2}$.

Let us find the partial fraction decomposition $\frac{6}{s^2(s+1)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}$. Then, clearing the denominator, this equation becomes $6 = (s^2+s)A+(s+1)B+s^2C$, which, regrouping like terms is $(A+C)s^2+(A+B)s+B = 6$. Thus $A+C = 0$, $A+B = 0$,and $B = 6$, which we can solve to yield $A = -6$ and $C = 6$. Thus $\frac{6}{s^2(s+1)} = \frac{6}{s^2} - \frac{6}{s} + \frac{6}{s+1}$, whose inverse Laplace transform is $6t - 6 + 6e^{-t}$.

6.4.3. *Find the inverse Laplace transform of the expression* $\frac{s+5}{s^2+6s+10}$.

The denominator is irreducible in real numbers (the roots of $s^2+6s+10$ are complex), so we must complete the square to find a form of which we can take the inverse Laplace transform. $s^2+6s+10 = (s+3)^2+1$, so we can rewrite the expression as $\frac{s+3}{(s+3)^2+1} + \frac{2}{(s+3)^2+1}$, which has inverse Laplace transform $e^{-3t}\cos t + 2e^{-3t}\sin t$.

6.4.9. *Use the Laplace transform to solve the initial value problem* $y'' + 4y = 16t$ *with homogeneous initial conditions at* $t = 0$.

Note that with homogeneous initial conditions at $t = 0$, $L(y^{(n)}) = s^n$, so applying the Laplace transform to this equation yields $s^2Y + 4Y = \frac{16}{s^2}$, so $Y = \frac{16}{s^2(s^2+4)}$. The partial fraction decomposition of this expression is $\frac{16}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2+4} + \frac{Ds}{s^2+4}$, which reduces on multiplication by the common denominator to $A(s^3+4s)+B(s^2+4)+Cs^2+Ds^3 = 16$. Thus, equating the cubic, quadratic, linear, and constant terms, we find that $A+D=0$, $B+C=0$, $4A=0$, and $4B=16$, so $A=D=0$, $B=4$, and $C=-4$, so $Y = \frac{4}{s^2} - \frac{4}{s^2+4}$, whose inverse Laplace transform is $y = 4t - 2\sin(2t)$.

6.4.11. *Use the Laplace transform to solve the initial value problem $y'' + 4y = 3\sin t$; $y(0) = 0$, $y'(0) = -1$.*

Applying the Laplace transform to this equation yields $s^2Y - sy(0) - y'(0) + 4Y = \frac{3}{s^2+1}$, which, supplied with initial value, is $s^2Y + 1 + 4Y = \frac{3}{s^2+1}$, so $Y = \frac{2-s^2}{(s^2+1)(s^2+4)}$. The partial fraction decomposition of this expression is $\frac{2-s^2}{(s^2+1)(s^2+4)} = \frac{A}{s^2+1} + \frac{Bs}{s^2+1} + \frac{C}{s^2+4} + \frac{Ds}{s^2+4}$, which reduces on multiplication by the common denominator to $2 - s^2 = A(s^2+4) + B(s^3+4s) + C(s^2+1) + D(s^3+s)$. Thus, equating the cubic, quadratic, linear, and constant terms, we find that $B+D=0$, $A+C=-1$, $4B+D=0$, and $4A+C=2$. Thus $A=1$, $B=D=0$, and $C=-2$, so $Y = \frac{1}{s^2+1} - \frac{2}{s^2+4}$, which has inverse Laplace transform $y = \sin t - \sin 2t$.

6.4.19. *Use the Laplace transform to solve the initial value problem*

$$\begin{cases} x' = x + y; & x(0) = 1 \\ y' = -x - y - t; & y(0) = 0 \end{cases}$$

Applying the Laplace transform to this system yields

$$\begin{cases} sX - 1 = X + Y \\ sY = -X - Y - \dfrac{1}{s^2} \end{cases}$$

which we may rewrite in the more conventional form

$$\begin{cases} (s-1)X - Y = 1 \\ X + (s+1)Y = -\dfrac{1}{s^2} \end{cases}$$

Multiplying the first equation by $s+1$ and adding to the second gives $s^2X = \frac{s^3+s^2-1}{s^2}$, so $X = \frac{s^3+s^2-1}{s^4} = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4}$ and, from the first equation, $Y = (s-1)X - 1 = \frac{-s^2-s+1}{s^4} = -\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s^4}$. Taking the inverse Laplace transform of each of these, $x = 1 + t - \frac{1}{6}t^3$ and $y = -t - \frac{1}{2}t^2 + \frac{1}{6}t^3$.

7.1.3. *Multiply the power series $(2t - t^3)\sum_{n=0}^{\infty} A_n t^{2n}$*

Simply multiplying by each term, we get $\sum_{n=0}^{\infty} 2A_n t^{2n+1} - A_n t^{2n+3}$, which we can separate and recombine as such:

$$\sum_{n=0}^{\infty} 2A_n t^{2n+1} - \sum_{n=0}^{\infty} A_n t^{2n+3} = \sum_{n=0}^{\infty} 2A_n t^{2n+1} - \sum_{n=1}^{\infty} A_{n-1} t^{2n+1} = A_0 t - \sum_{n=1}^{\infty}(2A_n - A_{n-1})t^{2n+1}$$

7.1.11. *Use index shifting to find a power series expansion for the expression $f''(t) - tf(t)$, where $f(t) = \sum_{n=0}^{\infty} A_n t^n$.*

$f''(t) - tf(t) = \sum_{n=2}^{\infty} n(n-1)A_n t^{n-2} - \sum_{n=0}^{\infty} A_n t^{n+1}$. We replace every occurrance of $n-2$ with n in the first sum, and every occurrance of $n+1$ with n in the second, to rephrase this expression as $\sum_{n=0}^{\infty}(n+2)(n+1)A_{n+2}t^n - \sum_{n=1}^{\infty} A_{n-1}t^n = 2A_2 + \sum_{n=1}^{\infty}((n+2)(n+1)A_{n+2} - A_{n-1})t^n$.

7.1.16. *Let $\mathcal{L}(y) = t^2y'' + 4ty' + 2y$. Give the series representation of*

$$\mathcal{L}\left(\sum_{n=0}^{\infty} A_n t^n\right).$$

Let $y = \sum_{n=0}^{\infty} A_n t^n$, so that the expression sought is simply $t^2y'' + 4ty' + 2y$ in series form. By the index-shifting lemma, these terms are $\sum_{n=2}^{\infty} n(n-1)A_n t^n + \sum_{n=1}^{\infty} 4nA_n t^n + \sum_{n=0}^{\infty} 2A_n t^n$, which, pulling out terms of degree less than 2, gives $2A_0 + 6A_1 + \sum_{n=2}^{\infty}(n(n-1) + 4n + 2)A_n t^n$.

7.2.3. *Use the ODE $y' + 2ty = 0$ to derive a power series expansion for $f(t) = e^{-t^2}$.*

Let $y = \sum_{n=0}^{\infty} A_n t^n$. Then $y' + 2ty = \sum_{n=0}^{\infty}(n+1)A_{n+1}t^n + \sum_{n=1}^{\infty} 2A_{n-1}t^n = A_1 + \sum_{n=1}^{\infty}((n+1)A_{n+1} + 2A_{n-1})t^n$. Since this must be equal to zero, it follows that the coefficient of each term is zero, so $A_1 = 0$ and $(n+1)A_{n+1} + 2A_{n-1} = 0$ for $n \geq 1$. We may rewrite the second condition as $A_{n+2} = \frac{-2A_n}{n+2}$ for $n \geq 0$, so it follows straightforwardly from the first condition that all odd terms A_{2n+1} are zero, while the even terms are governed by $A_{2n+2} = \frac{-A_{2n}}{n+1}$, which inductively we can show to be $\frac{(-1)^{n+1}}{(n+1)!}A_0$. So the general solution is $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_0 t^{2n}$; since the specific solution e^{-t^2} has constant term 1, since $A_0 = f(0)$, the series expansion of e^{-t^2} is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$.

7.2.15. *Find the solution of the initial value problem $y'' + 2ty' - y = 0$; $y(0) = 1$, $y'(0) = 0$ in the form of a power series. If it is feasible to do so, give the coefficients in closed form.*

Let $y = \sum_{n=0}^{\infty} A_n t^n$. Then $y'' + 2ty' - y = \sum_{n=0}^{\infty}(n+1)(n+2)A_{n+2}t^n + 2\sum_{n=1}^{\infty} nA_n t^n - \sum_{n=0}^{\infty} A_n t^n = (2A_2 - A_0) + \sum_{n=1}^{\infty}((n+1)(n+2)A_{n+2} + (2n-1)A_n$. Since this is equal to zero, each coefficient is equal to zero, so $2A_2 - A_0 = 0$ and $(n+1)(n+2)A_{n+2} + (2n-1)A_n = 0$ for $n \geq 1$. Thus $A_2 = \frac{1}{2}A_0$ and $A_{n+2} = \frac{1-2n}{(n+1)(n+2)}A_n$ for $n \geq 1$ (as luck would have it, this relation holds true for $n = 0$ too). From the initial conditions, $A_0 = 1$ and $A_1 = 0$; since A_{n+2} is a multiple of A_n, it thus follows that all odd-power coefficients A_{2n+1} are zero, while the even-terms we may derive through observation of the patterns: $A_0 = 1$, $A_2 = \frac{1}{2}$, $A_4 = \frac{1}{2} \cdot \frac{-3}{3\cdot 4}$, $A_6 = \frac{1(-3)}{4!} \cdot \frac{(-7)}{5\cdot 6}$, and in general, we find that for $n \geq 1$, $A_{2n} = \frac{(-1)^{n-1} 3\cdot 7\cdot 11\cdot\dots\cdot(4k-5)}{2n!}$, so

$$y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4k-5)}{2n!} t^{2n}$$

7.2.25. *Find the solution of the initial value problem $(1-t^2)y''-8ty'-12y=0$; $y(0)=1$, $y'(0)=0$ in the form of a power series. If it is feasible to do so, give the coefficients in closed form.*

Let $y=\sum_{n=0}^{\infty}A_nt^n$. Then

$$\begin{aligned}\mathcal{L}(y)&=\sum_{n=0}^{\infty}(n+2)(n+1)A_{n+2}t^n-\sum_{n=2}^{\infty}n(n-1)A_nt^n-8\sum_{n=1}^{\infty}nA_nt^n-12\sum_{n=0}^{\infty}A_nt^n\\&=(2A_2-12A_0)+(6A_3-8A_1-12A_1)t\\&+\sum_{n=(}^{\infty}(n+2)(n+1)A_{n+2}-(n(n-1)+8n+12)A_n)t^n\end{aligned}$$

Since this is equal to zero, each coefficient is equal to zero, so $A_2=6A_0$, $A_3=\frac{10}{3}A_1$, and $A_{n+2}=\frac{n^2+7n+12}{(n+2)(n+1)}A_n=\frac{2(n+3)(n+4)}{(n+2}n+1)A_n$ for $n>2$. From the initial conditions, $A_0=1$ and $A_1=0$, so using the recurrance, since A_{n+2} is a multiple of A_n, it is clear that every odd coefficient A_{2n+1} is zero. The even coefficients, on the other hand, are given by $A_0=1$, $A_2=6$, $A_4=\frac{5\cdot6}{3\cdot4}6=15$, and so forth. It can be easily inductively shown that $A_{2n}=\frac{1}{2}\frac{(2n+2)!}{(2n)!}=(n+1)(2n+1)$, so the solution is $y=\sum_{n=0}^{\infty}(n+1)(2n+1)t^{2n}$.

7.2.31. *Find a polynomial that satisfies the ODE $y''+ty'-6y=0$.*

Let $y=\sum_{n=0}^{\infty}A_nt^n$. Then

$$\begin{aligned}y''+ty'-6y&=\sum_{n=0}^{\infty}(n+2)(n+1)A_{n+2}t^n+\sum_{n=1}^{\infty}nA_nt^n-6\sum_{n=0}^{\infty}A_nt^n\\&=2A_2-6A_0+\sum_{n=1}^{\infty}((n+2)(n+1)A_{n+2}+(n-6)A_n)t^n\end{aligned}$$

Since this is equal to zero, each coefficient is equal to zero, so $A_2=3A_0$ and $A_{n+2}=\frac{(6-n)A_n}{(n+2)(n+1)}$. Note that this guarantees that $A_8=0$, and thus that A_{10}, A_{12}, etc. are zero. No such guarantee is madefor the odd coefficients, but we can force them to be zero by letting $A_1=0$; for simplicity's sake, let $A_0=1$. Then, while $0=A_1=A_3=A_5=A_7=A_8=A_9=\cdots$, the only nonzero coefficients of this series will be $A_0=1$, $A_2=\frac{6}{1\cdot2}=3$, $A_4=\frac{4}{3\cdot4}(-3)=1$, and $A_6=\frac{2}{5\cdot6}=\frac{1}{15}$. Thus such a polynomial is $y=1+3t^2+t^4+\frac{1}{15}t^6$.