Name and TA Section:	
Student Number:	

SAMPLE Final Exam, Math 20D, 2004 Fall Quarter

Place/Time: PCYNH/MULTI 106, 8:00-10:59am, 9 December 2004

Instructions: Please write your name and/or student number on each page of the exam, and then solve the following seven problems. If you need extra space, use the back of each page of the exam (in this case, clearly indicate which problem you are solving). (NOTE: Anything in red or blue would not appear on a real exam. Three questions appear below, covering material in HW8-HW10 which was Chapter 5-7 in Conrad. The remaining four questions on real final will come from HW1-HW7 and two midterms, covering Chapters 1-4 in Conrad and Chapter 11 in Stewart.)

Problem 5. (25 points) We are given the following second order ODE:

$$y'' - 6y' + 9y = te^{3t}.$$

(a) (10 points) Find the general solution $y_h(t)$ to the homogeneous problem.

SOLUTION: This is a second-order ODE of the form:

$$y'' + p(t)y' + q(t)y = g(t),$$

with $p(t) = -6$, $q(t) = 9$, and $g(t) = te^{3t}$. We first find the characteristic values by looking for a homogeneous solution in the form $y_h(t) = Ce^{st}$, and plug this into the homogeneous equation:

$$s^2Ce^{st} - 6sCe^{st} + 9Ce^{st} = 0.$$

Dividing out the common Ce^{st} we are left with the quadratic equation for s:

$$s^2 - 6s + 9 = (s-3)^2 = 0.$$

With repeated roots $s_1 = s_2 = 3$, we know that two linearly independent solutions are $y_1(t) = e^{3t}$ and $y_2(t) = te^{3t}$, and so the homogeneous solution is:

$$y_h(t) = C_1y_1(t) + C_2y_2(t) = C_1e^{3t} + C_2te^{3t}.$$

(b) (10 points) Find a particular solution to the inhomogeneous problem.

SOLUTION: We will, as usual, use variation of parameters to find a particular solution $y_p(t)$. Since $y_h(t) = C_1y_1(t) + C_2y_2(t)$, variation of parameters requires we look for a particular solution of the form:

$$y_p(t) = w_1(t)y_1(t) + w_2(t)y_2(t).$$

In order to determine $w_1(t)$ and $w_2(t)$, we have two options. We can either remember the complicated formulas for them given in Chapter 5 of Contrad, or we can just simply write the second order problem as a first-order system for finding the particular solution. This way the number of things we have to remember is actually quite small. Writing the problem as a first order system (setting $x = dy/dt$) gives:

$$\begin{aligned} y' &= x, \\ x' &= g(t) - q(t)y - p(t)x = te^{3t} - 9y + 6x, \end{aligned}$$

or as a matrix system $v' = Av + f$, with:

$$v = \begin{pmatrix} y(t) \\ x(t) \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ g(t) \end{pmatrix} = \begin{pmatrix} 0 \\ te^{3t} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix}.$$

Since we have the homogeneous solution in part (a), we know that the fundamental matrix solution for the homogeneous problem is:

$$\mathcal{X}(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t+1)e^{3t} \end{pmatrix}.$$

We now proceed exactly as usual, meaning we solve: $\mathcal{X}w' = f$, for $w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$, where $f(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix} = \begin{pmatrix} 0 \\ te^{3t} \end{pmatrix}$. The system is then:

$$\begin{pmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t+1)e^{3t} \end{pmatrix} \begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ te^{3t} \end{pmatrix}.$$

Solving this 2×2 system in the usual way (adding/subtracting multiples of the equations together) gives $w_1' = -t^2$ and $w_2' = t$, which then gives $w_1 = -\frac{1}{3}t^3$ and $w_2 = \frac{1}{2}t^2$. The particular solution is then:

$$y_p(t) = w_1(t)y_1(t) + w_2(t)y_2(t) = w_1(t)e^{3t} + w_2(t)te^{3t} = -\frac{1}{3}t^3e^{3t} + \frac{1}{2}t^2te^{3t} = \frac{1}{6}t^3e^{3t}.$$

(c) (5 points) Find the general solution.

SOLUTION: We can now simply write down the general solution: $y(t) = y_h(t) + y_p(t) = C_1e^{3t} + C_2te^{3t} + \frac{1}{6}t^3e^{3t}$.

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Problem 6. (25 points) We are given the following second order IVP:

$$y'' + y' + y = -2\sin t,\ y(0) = 2,\ y'(0) = 0.$$

(Just in case you need them (hint hint), recall that $\mathcal{L}[\sin(kt)](s) = \frac{k}{s^2+k^2}$ and $\mathcal{L}[\cos(kt)](s) = \frac{s}{s^2+k^2}$.)

(a) (10 points) Apply the Laplace Transform to both sides of the equation, producing an equation for $Y(s) = \mathcal{L}[y(t)](s)$.

SOLUTION: We first assemble the Laplace Transforms of the various terms appearing in the equation. Defining $Y(s) = \mathcal{L}[y(t)](s)$, we have:

$$\begin{aligned} \mathcal{L}[y'(t)](s) &= sY - y(0) = sY - 2 \\ \mathcal{L}[y''(t)](s) &= s^2Y - sy(0) - y'(0) = s^2Y - 2s \\ \mathcal{L}[\sin(t)](s) &= \frac{1}{s^2+1}. \end{aligned}$$

We can now take the transform of both sides of the equation, to yield the simpler equation for Y:

$$s^2Y - 2s + sY - s + Y = \frac{-2}{s^2+1}.$$

(b) (5 points) Solve the resulting equation for $Y(s)$.

SOLUTION: Collecting terms gives:

$$(s^2 + s + 1)Y - 2s - 2 = \frac{-2}{s^2+1}.$$

Rearranging this gives:

$$(s^2 + s + 1)Y = 2s + 2 + \frac{-2}{s^2+1} = 2\left(\frac{(s+1)(s^2+1)}{s^2+1} - \frac{1}{s^2+1}\right) = 2s\left(\frac{s^2+s+1}{s^2+1}\right).$$

This gives finally

$$Y = \frac{2s}{s^2+1}.$$

(c) (10 points) Apply the inverse Laplace Transform to both sides of the resulting equation to obtain the solution $y(t) = \mathcal{L}^{-1}[Y(s)](t)$.

SOLUTION: By observation we have clearly

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}[\frac{2s}{s^2+1}](t) = 2\cos(t).$$

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Problem 7. (25 points) We are given the following second order IVP:

$$y'' - ty' - (t^2+1)y = 0,\ y(0) = 2,\ y'(0) = 1.$$

(a) (10 points) Assume $y(t)$ is an analytic function, and substitute a series representation of $y(t)$ into the ODE.

SOLUTION: Let's assemble some of the terms we will need for substituting a series solution of the form:

$$y(t) = \sum_{n=0}^{\infty} C_n t^n$$

into the ODE. Besides $y(t)$ itself, the remaining terms are:

$$\begin{aligned} t^2 y(t) &= t^2 \sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} C_n t^{n+2} = \sum_{n=1}^{\infty} C_{n-1} t^{n+1} = \sum_{n=2}^{\infty} C_{n-2} t^n, \\ ty'(t) &= t \sum_{n=1}^{\infty} nC_n t^{n-1} = \sum_{n=1}^{\infty} nC_n t^n = \sum_{n=0}^{\infty} nC_n t^n, \\ y''(t) &= \sum_{n=2}^{\infty} n(n-1)C_n t^{n-2} = \sum_{n=1}^{\infty} (n+1)nC_{n+1} t^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} t^n. \end{aligned}$$

This gives the following (unsimplified) series expression once we substitute the series into the ODE:

$$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} t^n - \sum_{n=0}^{\infty} nC_n t^n - \sum_{n=2}^{\infty} C_{n-2} t^n - \sum_{n=0}^{\infty} C_n t^n = 0.$$

(b) (10 points) Determine the coefficients in the resulting series solution by solving (or at least simplifying) the appropriate recurrence relation for the coefficients.

SOLUTION: Collecting terms and using the notational convention $C_k = 0$ for $k < 0$ we have:

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)C_{n+2} - (n+1)C_n - C_{n-2}\right] t^n = 0.$$

This gives:

$$(n+2)(n+1)C_{n+2} - (n+1)C_n - C_{n-2} = 0,$$

or finally

$$C_{n+2} = \frac{(n+1)C_n + C_{n-2}}{(n+2)(n+1)},$$

for all $n > 0$. If we now use the initial conditions we have:

$$2 = y(0) = \sum_{n=0}^{\infty} C_n (0)^n = C_0,$$

$$1 = y'(0) = \sum_{n=1}^{\infty} nC_n (0)^{n-1} = C_1,$$

we have $C_0 = 2$ and $C_1 = 1$, and by convention $C_{-1} = 0$ and $C_{-2} = 0$. The recurrence relation gives then: $C_2 = 1$, $C_3 = 1/3$, and we can then write all of the rest of the coefficients as simply:

$$C_n = \frac{(n-1)C_{n-2} + C_{n-4}}{n(n-1)},\ n \geq 4.$$

(c) (5 points) Write down the final general series solution for $y(t)$; your result should not have any remaining unknown constants.

SOLUTION: The solution is just

$$y(t) = \sum_{n=0}^{\infty} C_n t^n,$$

with $C_0 = 2$, $C_1 = 1$, $C_2 = 1$, $C_3 = \frac{1}{3}$,

$$C_n = \frac{(n-1)C_{n-2} + C_{n-4}}{n(n-1)},\ n \geq 4.$$