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Student Number:	

Midterm #2, Math 20D, 2004 Fall Quarter

Place/Time: PCYNH/MULTI 106, 9:00-9:50am, 19 November 2004

Instructions: Please write your name and/or student number on each page of the exam, and then solve the following four problems. If you need extra space, use the back of each page of the exam (in this case, clearly indicate which problem you are solving).

Problem 1. (25 points) We are given the following IVP:

$$\begin{aligned} y' + ty &= t, \text{ on } (0, \infty), \\ y(0) &= 0. \end{aligned}$$

(a) (20 points) Find the general solution.

SOLUTION: This is a linear first-order inhomogenous ODE of the form: $a_1(t)y' + a_0(t)y = g(t)$, where $a_1(t) = 1$, $a_0(t) = t$, $g(t) = t$. We can use either an integrating factor, or find the homogeneous solution and then use variation of parameters. In either case, we will first write it in the form:

$$y' + p(t)y = f(t), \text{ where } p(t) = \frac{a_0(t)}{a_1(t)} = t, \; f(t) = \frac{g(t)}{a_1(t)} = t.$$

Integrating factor approach: We first compute the integrating factor

$$m(t) = e^{\int p(t)dt} = e^{\int t dt} = e^{t^2/2}.$$

We then have $[e^{t^2/2}y]' = [my]' = my' + mp(t)y = mf(t) = te^{t^2/2}$, so that $e^{t^2/2}y = e^{t^2/2} + C$, giving $y(t) = 1 + Ce^{-t^2/2}$.

Homogeneous solution plus variation of parameters approach: We first solve the homogeneous problem: $y_h' + p(t)y_h = 0$, which is

$$y_h(t) = Ce^{\int -p(t)dt} = Ce^{\int -tdt} = Ce^{-t^2/2}.$$

(As usual, we have $y_h(t) = 1/m(t)$, ignoring the constant.) We then find a particular solution in the form $y_p(t) = y_h(t)w(t)$. We know that $w(t)$ always satisfies the ODE:

$$y_h w' = f,$$

which comes from simply plugging $y_p(t)$ into the ODE $y' + p(t)y = f(t)$, and using the fact that $y_h(t)$ is the homogeneous solution. (Note if we work with $a_1(t)y' + a_0(t)y = g(t)$, then the ODE $w(t)$ satisfies is just slightly different: $a_1 y_h w' = g$; we get exactly the same result for $w(t)$.) Now, $e^{-t^2/2}w'(t) = y_h(t)w'(t) = f(t) = t$, giving $w' = te^{t^2/2}$, or $w(t) = e^{t^2/2}$. Finally then $y_p = y_h w = e^{-t^2/2}e^{t^2/2} = 1$, and so $y(t) = y_h(t) + y_p(t) = Ce^{-t^2/2} + 1$.

(b) (5 points) Find the specific solution corresponding to the given initial condition.

SOLUTION: We have that: $0 = y(0) = Ce^0 + 1 = C + 1$, so that $C = -1$, and thus the solution is $y(t) = 1 - e^{-t^2/2}$.

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Problem 2. (25 points) Let us assume that the dynamics of the bald eagle population around the world is accurately described by the standard *logistic growth with critical threshold model*:

$$\frac{dy}{dt} = -ry\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right),$$

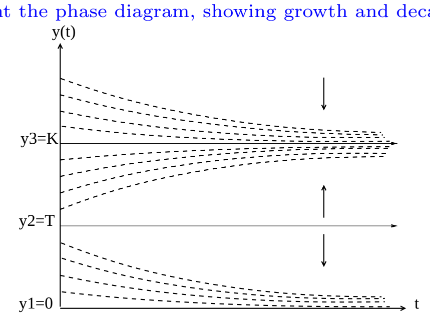
where $r > 0$ and $0 < T < K$. Here, $y(t)$ represents the biomass of the eagle population, r represents the ideal reproduction rate without environmental or other limitations, T represents the minimal population size needed to ensure genetic variability for continued reproduction, and K represents the maximal population supported by the current habitat (due to food and other limitations).

(a) (5 points) Determine the stationary solutions of the ODE.

SOLUTION: This autonomous ODE has the form: $y' = f(y)$. We can find stationary solutions of autonomous ODEs by simply finding y such that $f(y) = 0$ (since this implies $y' = 0$). Clearly, since $f(y)$ is a degree 3 polynomial in y, it has three roots, which by inspection are $y_1 = 0$, $y_2 = T$, and $y_3 = K$, with $0 = y_1 < y_2 < y_3 < +\infty$.

(b) (5 points) Draw the approximation solution plot and phase diagram in one picture, showing the asymptotic behavior of solutions that start with any non-negative initial data $y(0)$.

SOLUTION: We just need to show what happens to any solution in the intervals $0 = y_1 < y_2 < y_3 < +\infty$. Besides what we have already determined about location of the stationary solutions, the only other information we need is given by the ODE itself: $y' = f(y)$. I.e., for any $y \in (0, +\infty)$, $f(y)$ tells us when $y(t)$ is increasing or decreasing. For $y(t) \in (0, T)$ or $y(t) \in (K, +\infty)$, we find that $y' = f(y) < 0$, so that $y(t)$ is descreasing with increasing t. For $y(t) \in (T, K)$, we find that $y' = f(y) > 0$, so that $y(t)$ is increasing with increasing t. This information is shown in the following approximate solution plot and phase diagram. Solutions that start with various values for the initial condition are shown approximately in dotted lines; the vertical arrows represent the phase diagram, showing growth and decay of solutions for various ranges of $y(t)$.



(c) (5 points) Infer stability or instability of the stationary solutions by using only the picture in part (b). (It is hard to check stability directly for this problem due to the complicated form of $f(y)$ in the ODE, but the picture contains all the information we need.)

SOLUTION: The picture in part (b) clearly shows that the stationary solutions $y_1 = 0$ and $y_3 = K$ are stable, whereas the stationary solution $y_2 = T$ is not stable. This can be inferred from the arrows representing the phase diagram, or by the attraction of solutions to the stable stationary solutions y_1 and y_3, and the repulsion of solutions from the unstable stationary solution y_2.

(d) (5 points) Imagine that in year t, the population $y(t)$ drops below T due to disease. What can you say about longterm survivability of the bald eagle species? Justify your answer.

SOLUTION: Since $y'(t) = f(y) < 0$ when $0 < y(t) < T$, we know that the poplulation density $y(t)$ will decrease asymptotically to zero if at any point in time, $y(t)$ drops below T. In other words, the bald eagle would be doomed to extinction if the population ever drops below T.

(e) (5 points) Imagine the eagle habitat is reduced so that K is only slightly larger than T, for example, $K = 1.1T$. Why is this a very dangerous situation for the bald eagle species?

SOLUTION: Since the stable stationary solution is $y(t) = K$, we know that even if $y(0)$ starts out quite high, eventually the population will drop down close to $y(t) = K$. If at any time after that the population drops even slightly (by 10% or more) due to disease or other factors, the population would never recover, since $y(t)$ would drop below T.

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Problem 3. (25 points) Consider the following single second order ODE for $x(t)$:

$$x'' + x^2 = 0.$$

(a) (5 points) Derive the following equivalent first-order system of ODEs for $x(t)$ and $y(t)$:

$$\begin{aligned} x' &= y, \\ y' &= -x^2. \end{aligned}$$

SOLUTION: We simply define a new variable $y(t) = x'(t)$, giving the first equation above, and then substitute $y(t)$ into the original second order equation for any term involving $x'(t)$ or its derivatives:

$$(x')' + x^2 = y' + x^2 = 0,$$

which gives then the second equation above.

(b) (10 points) Write down the single ODE for the orbits of the system of ODEs in part (a).

SOLUTION: We know that the equation for the orbits is simply the equation for the curve $(x(t), y(t))$ that is traced out in the phase plane by the solution of the system of ODEs. If the original ODE system has the form:

$$\begin{aligned} x' &= f(x,y), \\ y' &= g(x,y), \end{aligned}$$

then differential equation for the orbits is obtained as:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x,y)}{f(x,y)}.$$

For this particular example, we have $g(x,y) = -x^2$ and $f(x,y) = y$, so that the orbit ODE is:

$$\frac{dy}{dx} = \frac{-x^2}{y}.$$

(c) (10 points) Find an integral $F(x,y) = C$ for the ODE system in part (a) by solving the orbit ODE.

SOLUTION: We just solve the single ODE for the orbits to produce $F(x,y)$. We will have trouble computing a general integrating factor $m(x,y)$ for the orbit ODE, since it will end up involving both variables x and y. However, it turns out to be separable, and as we learned in Problem 22 in Section 2.2, separation of variables can be viewed as the special case where you can actually determine $m(x,y)$.

The ODE for the orbits again has the form:

$$\frac{dy}{dx} = \frac{-x^2}{y}.$$

Separating, we have:

$$y dy + x^2 dx = 0.$$

We now seek $F(x,y)$ such that:

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = P(x,y)dx + Q(x,y)dy = 0, \tag{1}$$

where in this case, $P(x,y) = P(x) = x^2$, and $Q(x,y) = Q(y) = y$. Being separable, it is always exact since $\partial P/\partial y = 0 = \partial Q/\partial x$. Since $P(x,y) = P(x)$ and $Q(x,y) = Q(y)$, we know any $F(x,y)$ satisfying (1) must satisfy both $F(x,y) = \int P(x)dx + v(y)$ and $F(x,y) = \int Q(y)dy + u(x)$; one can verify this by computing $\partial F/\partial x$ and $\partial F/\partial y$. This requires $u(x) = \int P(x)dx$ and $v(y) = \int Q(y)dy$, so $F(x,y) = \int P(x)dx + \int Q(y)dy$. Computing these simple integrals gives $F(x,y) = \frac{1}{3}x^3 + \frac{1}{2}y^2$, which gives then finally:

$$F(x,y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 = C.$$

Problem 1	
Problem 2	
Problem 3	
Problem 4	
TOTAL:	

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Problem 4. (25 points) We are given the following ODE system: $v' = Av + f$, where

$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad f = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad v(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

(a) (10 points) Find the fundamental solution matrix $\mathcal{X}(t)$ and homogeneous solution $v_h = \mathcal{X}c$.

SOLUTION: We look for the fundamental solutions $v_1(t) = e^{r_1 t} b_1$, $v_2(t) = e^{r_2 t} b_2$, where (r_k, b_k), $k = 1, 2$ are the two eigenpairs of A. We first determine the eigenvalues of A by solving the quadratic equation obtained as $\det(A - sI) = s^2 - \operatorname{tr}(A)s + \det(A) = 0$, which in this case is

$$s^2 - 3s + 2 = (s-1)(s-2) = 0.$$

The roots are then $s_1 = 1$ and $s_2 = 2$. We now find the eigenvectors.

$$\boxed{s_1 = 1} : Ab_1 = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = s_1 b_1 = \begin{pmatrix} h \\ k \end{pmatrix},$$

which gives $3h + k = 2h$ and $h + 3k = 2k$, either of which give $h = -k$. Thus, $b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\boxed{s_2 = 2} : Ab_2 = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = s_2 b_2 = 2 \begin{pmatrix} h \\ k \end{pmatrix},$$

which gives $3h + k = 4h$ and $h + 3k = 4k$, either of which give $h = k$. Thus, $b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This gives then $v_1(t) = e^{r_1 t} b_1 = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $v_2(t) = e^{r_2 t} b_2 = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Defining now the fundamental solution matrix $\mathcal{X}(t)$ as

$$\mathcal{X}(t) = \begin{pmatrix} v_1(t) & v_2(t) \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} \\ -e^t & e^{2t} \end{pmatrix},$$

we have that $v_h = \mathcal{X}(t)c$, with $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

(b) (10 points) Find a particular solution $v_p = \mathcal{X}(t)w(t)$ using variation of parameters.

SOLUTION: There is no reason to rederive the formula for $w(t)$ here (unless we cannot remember it, in which case we would rederive it). We know that doing variation of parameters (plugging v_p back into the ODE) will always produce the following ODE for the unknown $w(t)$:

$$\mathcal{X}(t)w' = f(t), \text{ which is : } \begin{pmatrix} e^t & e^{2t} \\ -e^t & e^{2t} \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

We must solve for $w_1(t)$ and $w_2(t)$. Adding the two equations together gives $2e^{2t} w_2' = 2e^t$, or $w_2' = e^{-t}$, so that $w_2(t) = -e^{-t}$. We now plug this into the first equation to get $e^t w_1' + e^{2t} e^{-t} = e^t$, or $w_1' = 0$, giving $w_1(t) = C$, for any constant C; we will take $C = 0$, so that $w_1(t) = 0$. Thus, $v_p(t) = \mathcal{X}(t)w(t)$, with $w(t) = \begin{pmatrix} 0 \\ -e^{-t} \end{pmatrix}$.

(c) (5 points) Find the solution $v(t)$ corresponding to the given initial data $v(0)$.

SOLUTION: Just write down the final general solution in the form $v(t) = v_h(t) + v_p(t) = \mathcal{X}(t)(c + w(t))$ and then use the initial data $v(0)$ to determine c. I.e., $v(0) = \mathcal{X}(0)(c + w(0))$, which is

$$v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^0 & e^{2(0)} \\ -e^0 & e^{2(0)} \end{pmatrix} \left\{ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -e^0 \end{pmatrix} \right\} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 - 1 \end{pmatrix},$$

which gives $c_1 + (c_2 - 1) = 0$ and $-c_1 + (c_2 - 1) = 1$, or $c_1 + c_2 = 1$ and $c_2 - c_1 = 2$, giving $c_2 = 3/2$ and $c_1 = -1/2$. This gives then finally $v(t) = \mathcal{X}(t)(c + w(t))$, where $c = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$. To verify we have it right, compute $v'(t)$, and verify that it is the same as $Av(t) + f(t)$.