$\boxed{4.1.1}$ (a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Solve normal eqn $A^TA\bar{x} = A^Tb$

where $A^TA = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$\Rightarrow \bar{x} = \begin{bmatrix} -1/7 \\ 10/7 \end{bmatrix}$

$\Rightarrow b - A\bar{x} = \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} & -\frac{1}{7} \end{bmatrix}^T \Rightarrow \|r\|_2 = \sqrt{14}/7.$

(b). $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Solve normal eqn $A^TA\bar{x} = A^Tb$

where $A^TA = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$\Rightarrow \bar{x} = \begin{bmatrix} -1/2 \\ 2 \end{bmatrix}$

$\Rightarrow b - A\bar{x} = \begin{bmatrix} -1/2 & 1 & -1/2 \end{bmatrix}^T \Rightarrow \|r\|_2 = \sqrt{3/2}.$

HW4

$\boxed{4.1.5}$ Proof: (i) Check $\|x\|_2 = 0$ iff $x=0$

"If": if $x=0$, then $\sqrt{\sum_i x_i^2} = 0$

"only if": if $\sqrt{\sum_i x_i^2} = 0 \Rightarrow$ each $x_i = 0$ for all i

$\Rightarrow x = 0$.

(ii) Check $\|kx\|_2 = |k|\|x\|_2$ for $\forall k \in \mathbb{R}$

$$\|kx\|_2 = \sqrt{\sum_i (k x_i)^2} = \sqrt{k^2 \sum_i x_i^2} = |k| \sqrt{\sum_i x_i^2} = |k| \, \|x\|_2.$$

(iii) Check $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$

Take square on both sides

We have $\|x+y\|_2^2 = \sum_i x_i^2 + \sum_i y_i^2 + 2\sum_i |x_i y_i|$

by Cauchy-Schwarz

$$\leq \sum_i x_i^2 + \sum_i y_i^2 + 2\|x\|_2 \|y\|_2 = (\|x\|_2 + \|y\|_2)^2$$

Thus, $\|\cdot\|_2$ is a vector norm. $\square$.

$\boxed{4.1.6}$ Proof: (a) Since $A^T \cdot (A^{-1})^T = A^{-1} A = I$

$(A^{-1})^T \cdot A^T = A \cdot A^{-1} = I$

Hence, $(A^T)^{-1} = (A^{-1})^T$.

(b) $Ax = b \Rightarrow b - Ax = 0$, where x minimize the residual

$\Rightarrow x$ solves the normal equs.

$\square$

$\boxed{5.1.1}$ (a) $h = 0.1$

Use Two-point forward-difference

$$f'(1) \approx \frac{f(1+0.1) - f(1)}{0.1} = 0.9531$$

$$\text{Error} = |f'(1) - 0.9531| = |1 - 0.9531| = 0.0469.$$

$\boxed{5.1.5}$ Use Three-point centered-difference for $f''(x)$

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

$$\text{(a). } f''(1) \approx \frac{f(0.9) - 2f(1) + f(1.1)}{0.01} = 2.0202$$

$$\text{Error} = |f''(1) - 2.0702| = |2 - 2.0202| = 0.0202.$$

$\boxed{5.1.11}$ Apply (5.15)

$$Q \approx \frac{2^n F(h/2) - F(h)}{2^n - 1}, \quad \text{where } F(h) = \frac{f(x+h) - f(x)}{h}, \quad n = 1$$

$$\Rightarrow Q \approx 2\left(\frac{f(x+h/2) - f(x)}{h/2}\right) - \frac{f(x+h) - f(x)}{h}$$

$$= \frac{4f(x+h/2) - 3f(x) - f(x+h)}{h}$$

$\boxed{5.1.21}$ Proof: Use Three-point centered-difference formula for $f^{(4)}(x)$, one has

$$f^{(4)}(x) = \frac{f''(x-h) - 2f''(x) + f''(x+h)}{h^2} - \frac{h^2}{12} f^{(6)}(c)$$

Substitute $f''(x-h)$, $f''(x)$, $f''(x+h)$ by Three-pts Centered-diff,

$$f^{(4)}(x) = \left(\frac{f(x-2h) - 2f(x-h) + f(x)}{h^4}\right) + (-2)\left(\frac{f(x-h) - 2f(x) + f(x+h)}{h^4}\right)$$
$$+ \left(\frac{f(x) - 2f(x+h) + f(x+2h)}{h^4}\right) + O(h^2)$$

$$= \frac{f(x-2h) - 4f(x-h) + 6f(x) - 4f(x+h) + f(x+2h)}{h^4} + O(h^2)$$

$\square$.

$\boxed{5.2.1}$ Composite Trapezoid Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i\right)$$

(c) $\int_0^1 e^x dx \approx \frac{1}{2}(1 + e) = 1.8591$ (m=1)

error $= \left|\int_0^1 e^x dx - 1.8591\right| = 0.149$

(m=2) $\int_0^1 e^x dx \approx \frac{1}{4}(1 + e + 2e^{0.5}) = 1.7539$

error $= \left|\int_0^1 e^x dx - 1.7539\right| = 0.0356$

$(m=4)$ $\int_0^1 e^x dx \approx \frac{1}{8}(1+e+2e^{\frac{1}{4}}+2e^{\frac{2}{4}} + 2e^{3/4}) = 1.7272$ (5)

$\text{error} = \left|\int_0^1 e^x dx - 1.72721\right| = 0.0089$

$\boxed{5.2.3}$ Composite Simpson's Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left(y_0 + y_{2m} + 4\sum_{i=1}^{m} y_{2i-1} + 2\sum_{i=1}^{m-1} y_{2i}\right)$$

(c). $(m=1)$ $\int_0^1 e^x dx \approx \frac{1}{6}(1+e+4e^{\frac{1}{2}}) = 1.7189$

$\text{error} = \left|\int_0^1 e^x dx - 1.7189\right| = 5.79\times 10^{-4}$

$(m=2)$ $\int_0^1 e^x dx \approx \frac{1}{12}(1+e+4e^{\frac{1}{4}}+4e^{\frac{3}{4}}+2e^{\frac{2}{4}})$
$= 1.7183$

$\text{error} = \left|\int_0^1 e^x dx - 1.7183\right| = 3.7\times 10^{-5}$

$(m=4)$ $\int_0^1 e^x dx \approx \frac{1}{24}\left(1+e+4(e^{\frac{1}{8}}+e^{\frac{3}{8}}+e^{\frac{5}{8}}+e^{\frac{7}{8}}) + 2(e^{\frac{2}{8}}+e^{\frac{4}{8}}+e^{\frac{6}{8}})\right)$

$= 1.71828$

$\text{error} = \left|\int_0^1 e^x dx - 1.71828\right| = 2.33\times 10^{-6}$

$\boxed{5.2.9}$ Simpson's Rule

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(y_0+4y_1+y_2)$$

$$\Rightarrow \int_0^1 x^4 dx \approx \frac{1}{6}\left(0+4\left(\tfrac{1}{2}\right)^4+1\right) = \cancel{0.1771}\ 0.2083$$

$$\Rightarrow \text{error} = \left|\int_0^1 x^4 dx - \cancel{0.1771}\ 0.2083\right| = \cancel{0.0229}\ 0.0083$$

Look at the error term in (5.22)

$$R(x) = \frac{h^5}{90}f^{(4)}(c) = \frac{h^5}{90}\cdot 24 = 0.0083$$

which matches the exact error.

$\boxed{5.2.12}$ Refer to the defination of degree of precision, this method must be exatly for all polynomials of degree 0, 1, 2.

Then, we plug $f(x)=1, x, x^2$ into this method,

we have

$$\int_0^1 1\,dx = 1 = C_1+C_2+C_3$$

$$\int_0^1 x\,dx = \frac{1}{2} = 0+\frac{1}{2}C_2+C_3$$

$$\int_0^1 x^2 dx = \frac{1}{3} = 0+\frac{1}{4}C_2+C_3$$

, solve these three equations with three unknowns,

we have $\begin{cases} C_1 = \frac{1}{6} \\ C_2 = \frac{2}{3} \\ C_3 = \frac{1}{6} \end{cases} \Rightarrow \int_0^1 f(x)dx \approx \frac{1}{6}\left(f(0)+4f\left(\tfrac{1}{2}\right)+f(1)\right)$

which is just Simpson's Rule.

$\boxed{5.3.1}$ (c) $\int_0^1 e^x dx$.

$R_{11} = (b-a)\frac{f(a)+f(b)}{2} = \frac{1+e}{2} = 1.8591$

According to eq ~~(5.3)~~ (5.31)

One has $R_{21} = \frac{1}{2}R_{11} + \frac{b-a}{2} f(\frac{a+b}{2})$, $R_{31} = \frac{1}{2}R_{21} + \frac{1}{4}(b-a)\cdot(f(\frac{1}{4})+f(\frac{3}{4}))$

$= 1.7539$ $\qquad = 1.7272$

According to eq (5.35)

One has $R_{22} = \frac{2^2 R_{21} - R_{11}}{3} = 1.7188$

$R_{32} = \frac{2^2 R_{31} - R_{21}}{3} = 1.7183$

$R_{33} = \frac{4^2 R_{32} - R_{22}}{4^2-1} = 1.71827$

$\boxed{5.3.3}$ Notice that

$R_{11} = \frac{b-a}{2}(f(a)+f(b))$ is Composite Trapezoid Rule with $h=h_1$

$R_{21} = \frac{(b-a)}{4}(f(a)+f(b)+2f(\frac{a+b}{2}))$ is composite Trapezoid Rule with $h=h_2$

$\Rightarrow R_{22} = \frac{2^2 R_{21} - R_{11}}{3} = \frac{(b-a)}{6}(f(a)+f(b)+4f(\frac{a+b}{2}))$

is just Composite Simpson's Rule with $h=h_2$.

$\boxed{5.5.1}$

(a) According to Table 5.1

The Gaussian Quadrature of $n=2$

is $\int_{-1}^{1} f(x)dx \approx f(-\sqrt{\frac{1}{3}}) + f(\sqrt{\frac{1}{3}})$

thus: $\int_{-1}^{1}(x^3+2x)dx \approx (-\sqrt{\frac{1}{3}})^3+2(-\sqrt{\frac{1}{3}}) + (\sqrt{\frac{1}{3}})^3+2\sqrt{\frac{1}{3}}$

$=0$

error $= \left|\int_{-1}^{1}(x^3+2x)dx - 0\right| = 0$

$\boxed{5.5.7}$ Proof:

Only need to check $\int_{-1}^{1} P_1(x)P_2(x)\,dx = 0$ or not

$\int_{-1}^{1} x(x^2-\frac{1}{3})\,dx$

$= \left(\frac{1}{4}x^4 - \frac{1}{6}x^2\right)\Big|_{-1}^{1} = 0$,

thus, P_1 and P_2 are orthogonal to each other.

$\square$