MATH 174 HW3

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$\boxed{3.1.1}$ (a). $x_0 = 0$, $x_1 = 2$, $x_2 = 3$
$y_0 = 1$, $y_1 = 3$, $y_2 = 0$

$$\Rightarrow P_2(x) = y_0 \cdot \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= 1 \cdot \frac{(x-2)(x-3)}{6} + 3\frac{x(x-3)}{-2} + 0$$

$$= -\tfrac{4}{3}x^2 + \tfrac{11}{3}x + 1$$

(c) $x_0 = 0$, $x_1 = 2$, $x_2 = 4$
$y_0 = -2$, $y_1 = 1$, $y_2 = 4$.

$\Rightarrow P_2(x) =$ same formula as Part (a)

$$= (-2)\frac{(x-2)(x-4)}{8} + \frac{x(x-4)}{-4} + 4\frac{x(x-2)}{8}$$

$$= \tfrac{3}{2}x - 2$$

$\boxed{3.1.2}$ (a) $P_2(x) = f[x_0] + f[x_0\ x_1](x-x_0) + f[x_0\ x_1\ x_2](x-x_0)(x-x_1)$

where
$0 = x_0$ | $f[x_0] = 1$ > $f[x_0\ x_1] = \dfrac{f[x_1]-f[x_0]}{x_1-x_0} = 1$ > $f[x_0\ x_1\ x_2] = -4/3$
$2 = x_1$ | $f[x_1] = 3$ > $f[x_1 . x_2] = \dfrac{f[x_2]-f[x_1]}{x_2-x_1} = -3$
$3 = x_2$ | $f[x_2] = 0$

$$\Rightarrow P_2(x) = 1 + (x-0) + (-\tfrac{4}{3})(x-0)(x-2)$$

$= -\tfrac{4}{3}x^2 + \tfrac{11}{3}x + 1$ same as result derived from ~~Leg~~ Lagrange.

(c) $P_1(X)$ = same formula as (a) ②

where

$$\begin{array}{c|cccc} x_0=0 & f[x_0]=-2 & & & \\ & & f[x_0,x_1]=\frac{3}{2} & & \\ x_1=2 & f[x_1]=1 & & f[x_0,x_1,x_2]=0 & \\ & & f[x_1,x_2]=\frac{3}{2} & & \\ x_2=4 & f[x_2]=4 & & & \end{array}$$

$\Rightarrow P_1(X) = -2 + \frac{3}{2}(X-0) + 0$

$= \frac{3}{2}X - 2$, same as result derived from Lagrange.

$\boxed{3.1.3}$ By Thm 3.2, there must be only one Polynomial of degree less than 4 for there 4 given points.

Let's find this only Polynomial first by Newton's dividsion.

$P(X) = f[x_0] + f[x_0,x_1](X-x_0) + f[x_0,x_1,x_2](X-x_0)(X-x_1)$
$+ f[x_0,x_1,x_2,x_3](X-x_0)(X-x_1)(X-x_2)$

where

$$\begin{array}{c|ccccccc} x & f[\,] & & & & & & \\ -1 & 3 & & & & & & \\ & & -1 & & & & & \\ 1 & 1 & & 1 & & & & \\ & & 2 & & 0 & & & \\ 2 & 3 & & 1 & & & & \\ & & 4 & & & & & \\ 3 & 7 & & & & & & \end{array}$$

$\Rightarrow$ Since $f[x_0,x_1,x_2,x_3]=0$, $P(x)$ has degree 2

$\Rightarrow$ (a) For $d=2$, there is only one Polynomial,

$P_2(x) = 3 - (X+1) + (X+1)(X-1) = X^2 - X + 1$

(b). There is no Polynomial at degree $d=3$

(c) There are infinit many Polynomials at degree $d=6$ ③

Here is one example.

$P_6(x) = $ ~~[illegible]~~

$P_2(x) + x^2(x+1)(x-1)(x-2)(x-3)$

$\boxed{3.1.11}$ show, By Thm 3.2, for 4 points,

there is only One Polynomial at degree ~~less~~ Less than 4.

Now, there is one $y = ax^2 + bx + c$ passing through these 4 points,

Thus, it is the only one with degree less than 4.

Hence, there is $\boxed{\text{no polynomials}}$ at degree 3 passing through these 4 points.

$\boxed{3.2.1}$ (a). Use Newton's form.

x	$f[\,]\cdots$		
0	0		
		$\frac{2}{\pi}$	
$\pi/2$	1		$-\frac{4}{\pi^2}$
		$-\frac{2}{\pi}$	
π	0		

$$\Rightarrow P_2(x) = 0 + \frac{2}{\pi}(x-0) - \frac{4}{\pi^2}(x-0)(x-\frac{\pi}{2})$$

$$= -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x$$

(b). $P_2(\pi/4) = \frac{3}{4}$

(c) By Thm 3.4

$$|f(x)-P_2(x)| = \left|\frac{x(x-\frac{\pi}{2})(x-\pi)}{3!}f'''(c)\right|, \quad c\in[0,\pi], \; x\in[0,\pi]$$

$$= \left|\frac{x(x-\frac{\pi}{2})(x-\pi)}{6}\cos c\right|$$

Pick $x=\pi/4$

$$\le \left|\frac{\frac{\pi}{4}(\frac{\pi}{4}-\frac{\pi}{2})(\frac{\pi}{4}-\pi)}{6}\cdot 1\right|$$

$$= 0.2422$$

(d). $\sin(\pi/4) = \sqrt{2}/2$

$\Rightarrow$ Actual Error $= |\sin\frac{\pi}{4} - P_2(\frac{\pi}{4})| = |\frac{\sqrt{2}}{2} - \frac{3}{4}| \approx 0.0429$

which is bounded by Error Bound in (c).

$\boxed{3.2.3}$ (a) By Thm 3.4

$$|f(\tfrac{1}{2}) - P_9(\tfrac{1}{2})| = \left|\frac{\sum_{i=0}^{9}(\frac{1}{2}-\frac{i}{9})}{10!}f^{(10)}(c)\right|, \quad c\in[0,1]$$

where $f^{(10)}(c) = e^{-2c}(-2)^{10}$

$$\Rightarrow |f(\tfrac{1}{2}) - P_9(\tfrac{1}{2})| \le \left|\frac{\sum_{i=0}^{9}(\frac{1}{2}-\frac{i}{9})}{10!}\cdot 2^{10}\right| \approx 7.0579\times 10^{-11}$$

(b). Notice that $P(\frac{1}{2}) \approx e^{-1}$, then $\frac{1}{P(\frac{1}{2})} \approx e$, and $e^{-2} < P(\frac{1}{2}) < 1$

$$\Rightarrow \left|e - \frac{1}{P(\frac{1}{2})}\right| = \left|\frac{e\cdot P(\frac{1}{2}) - 1}{P(\frac{1}{2})}\right| = \left|\frac{P(\frac{1}{2}) - e^{-1}}{P(\frac{1}{2})e^{-1}}\right| \le e^{3}|P(\tfrac{1}{2}) - e^{-1}|$$

$\approx 1.42\times 10^{-9} \le 0.5\times 10^{-8}$, i.e. At least 8 decimal places.

$\boxed{3.2.5}$ Since Error $= \left| \dfrac{\sum_{i=1}^{6} (x - i/10)}{6!} f^{(6)}(c) \right|$,

the only different for $x = 0.35$ or 0.55

is $\sum_{i=1}^{6} (x - i/10) \triangleq g(x)$

Since $g(0.35) = \cancel{0.35}\ 0.05^2 \cdot 0.15^2 \cdot 0.25^2$

$g(0.55) = 0.05^2 \cdot 0.15 \cdot 0.25 \cdot 0.35 \cdot 0.45$

$> g(0.35)$.

So, we expect the error at 0.35 is smaller.

$\boxed{3.3.1}$ (a). $[-1, 1]$, $n = 6$

The Chebyshev nodes are

$$x_i = \frac{b+a}{2} + \frac{b-a}{2} \cos \frac{(2i-1)\pi}{2n}$$

$\Rightarrow \cancel{x_i}\ x_i = \cos \dfrac{(2i-1)\pi}{12}$, $i = 1, \cdots 6$

$\Rightarrow \left(\cos \frac{\pi}{12}, \cos \frac{3\pi}{12}, \cos \frac{5\pi}{12}, \cos \frac{7\pi}{12}, \cos \frac{9\pi}{12}, \cos \frac{11\pi}{12} \right)$.

(c). $[4, 12]$, $n = 6$.

$x_i = 8 + 4 \cos \dfrac{(2i-1)\pi}{12}$

$\Rightarrow \left(8 + 4\cos \frac{\pi}{12}, 8 + 4\cos \frac{3\pi}{12}, 8 + 4\cos \frac{5\pi}{12}, 8 + 4\cos \frac{7\pi}{12}, 8 + 4\cos \frac{9\pi}{12}, 8 + 4\cos \frac{11\pi}{12} \right)$.

(6)

$\boxed{3.3.2}$ (a) Simply apply Thm 3.6

One has $|(x-x_1)\cdots(x-x_n)| \le \frac{1}{2^{n-1}} = \frac{1}{2^5}$.

(c) Apply (3.14)

$$|(x-x_1)\cdots(x-x_n)| \le \frac{\left(\frac{b-a}{2}\right)^n}{2^{n-1}} = \frac{4^6}{2^5}.$$

$\boxed{3.4.1}$ Need to check (i) $S_1(1)=S_2(1)$

(ii) $S_1'(1)=S_2'(1)$

(iii) $S_1''(1)=S_2''(1)$

(a) $S_1'(1)=4$

$S_2'(1)=3\neq S_1'(1)$

So, not a cubic spline.

(b) $S_1(1)=S_2(1)=12$

$S_1'(1)=S_2'(1)=12$ $\Rightarrow$ it is a cubic spline.

$S_1''(1)=S_2''(1)=14$

$\boxed{3.4.2}$ (a) $S_1(1)=S_2(1)=10$

$S_1'(1)=S_2'(1)=20$ $\Rightarrow$ It's a cubic spline.

$S_1''(1)=S_2''(1)=30$

(b) Natural, Parabolically terminated, Not-a-knot

$S_1''(0)=6\neq 0$ — Not Natural.

$S_1(x)$ is degree 3 — Not Parabolically terminated.

$S_1'''(1)=S_2'''(1)=24$ — Yes, it is Not-a-knot.

$\boxed{3.4.4}$ (a) Find k_1, k_2, k_3

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From $S_1(1) = S_2(1) \Rightarrow 4 + k_1 + 2 - \frac{1}{6} = 1$

From $S_2'(2) = S_3'(2) \Rightarrow -\frac{4}{3} + 2k_2 - \frac{1}{2} = k_3$

From $S_2''(2) = S_3''(2) \Rightarrow 2k_2 - 1 = 2$

$\Rightarrow$ Solve Three equations with Three unknowns.

$$\begin{cases} k_1 = -\frac{29}{6} \\ k_2 = 3/2 \\ k_3 = 7/6 \end{cases}$$

(b). Natural: $S_1''(0) \neq 0$, Not Natural.

Parabolically terminated; S_1 is degree 3.

Not Parabolically terminated

Not-a-Knot: $S_1'''(1) = S_2'''(1)$

$S_2'''(2) = S_3'''(2)$

Yes, it's Not-a-Knot.