

Mid-Term I

①

+10 (a) $f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots + \frac{f^{(n)}(x)}{n!}h^n$

Or $= \sum_{i=0}^{\infty} \frac{f^{(i)}(x)}{i!} h^i$

+10 (b) $f(x+h) = f(x) + f'(x)h + \dots + \underbrace{\frac{f^{(n-1)}(x)}{(n-1)!}h^{n-1}}_{n \text{ terms}} + \frac{f^{(n)}(\xi)}{n!}h^n$, where $\xi \in (x, x+h)$.

2. (a) Let $f(x) = x^4 - x^3 - 10$,

+6 Then since $f(2) = -2 < 0$, $f(3) = 44 > 0$

Thus in $[2, 3]$, there exists a solution.

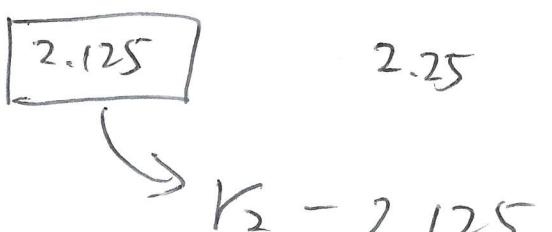
+7 (b)

a_n	$f(a_n)$	c	$f(c)$	b	$f(b)$
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Step 0: 2 -2 2.5 13.4375 3 44

Step 1: 2 -2 2.25 4.2383 2.5 13.4375

Step 2: 2 -2 2.125 2.25 4.2383



$$K_2 = 2.125.$$

+7 (c) Solve Ineq: $\frac{b-a}{2^{n+1}} = \frac{1}{2^{n+1}} \leq 10^{-10}$

$$\Rightarrow 2^{n+1} > 10^{10}$$

$$\Rightarrow (n+1)\log 2 > 10$$

$$\Rightarrow n > \frac{10}{\log 2} - 1 \approx 32.22$$

So totally
→ 33 steps are required.

3. (a) Truncate ~~the~~ Taylor series as following (2)

+5 Let r be the root, i.e. $f(r)=0$

$$0 = f(r) = f[x + (r-x)] = f(x) + f'(x)(r-x) + \dots$$

$$\Rightarrow r-x = -\frac{f(x)}{f'(x)}, \text{ def } r-x=h,$$

$$\text{then } r=x+h, \text{ where } h=-\frac{f(x)}{f'(x)}.$$

(b). $f(x)=x^2-x-1, \quad f'(x)=2x-1, \quad x_0=0$

+5 Step 1: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-1}{-1} = -1$

Step 2: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1 - \frac{1}{-3} = -\frac{2}{3}$.

(c) $x = x^2-1, \text{ where } g(x)=x^2-1.$

+5 (d) $x_1 = x_0^2-1 = -1$

+5 $x_2 = x_1^2-1 = 0$

4. $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$

i) $\|A\|_\infty = \max_i \left\{ \sum_{j=1}^2 |a_{ij}| \right\}, \quad j=1,2$

+4 $= 2$

ii) Since $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \Rightarrow \|A^{-1}\|_\infty = \frac{3}{2}$

+8 $\Rightarrow \text{Cond}(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = 3$

(3)

(iii) Eigenvalues.

$$+6 \quad \text{Solve } \det(A - \lambda I) = 0$$

$$\text{One has } (\lambda-2)(\lambda-1) = 0$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = 1$$

(iv) Spectral radius

$$+2 \quad R(A) = \max_i |\lambda_i| = 2$$

5. Jacobi: $X_{k+1} = D^{-1}(b - (L+U)X_k)$, where $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\stackrel{+10}{\Rightarrow} X_1 = D^{-1}(b - (L+U)X_0)$$

$$= D^{-1}b = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow X_2 = D^{-1}(b - (L+U)X_1)$$

$$= \cancel{\begin{bmatrix} 3 \\ -2 \end{bmatrix}} \quad \begin{bmatrix} 1 \\ -1/6 \end{bmatrix}$$

GS: ~~$\cancel{X_{k+1}}$~~ $= (D+L)X_{k+1} = b - UX_k$

$$\stackrel{+6}{\Rightarrow} X_1 = \begin{bmatrix} 3 \\ -1/6 \end{bmatrix}$$

$$\Rightarrow X_2 = \begin{bmatrix} 13/8 \\ -85/36 \end{bmatrix}$$