Math 171B: Numerical Optimization: Nonlinear Problems

Instructor: Michael Holst

Spring Quarter 2015

Homework Assignment #5
Due (See Class Webpage for Due Date)

In this homework we study techniques for constrained optimization. The starred exercises are those that require the use of Matlab. *You must do the* Matlab *problems to get credit for the homework.*

Exercise 5.1. Consider the nonlinearly constrained problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{minimize}} & 3x_2 + x_1^2 + x_2^2 \\ \text{subject to} & x_1^2 + (x_2+1)^2 - 1 = 0. \end{array} \tag{5.1}$$

(a) Show that $x(\alpha) = (\sin\alpha,\, \cos\alpha - 1)^T$ is a feasible path for the nonlinear constraint $x_1^2 + (x_2+1)^2 - 1 = 0$ of problem (5.1). Compute the tangent to the feasible path at $\bar{x} = (0,\, 0)^T$.

(b) If $f(x)$ denotes the objective function of problem (5.1), find an expression for $f\left(x(\alpha)\right)$ and compute $f\left(x(0)\right)$.

(c) Define the *Lagrangian function* $L(x,\lambda)$ and *constraint Jacobian* $J(x)$ for problem (5.1). Derive $\nabla L(x,\lambda)$, the gradient of the Lagrangian, and $\nabla^2_{xx} L(x,\lambda)$, the Hessian of the Lagrangian with respect to x.

(d) Determine whether or not the point $\bar{x} = (0,\, 0)^T$ is a constrained minimizer of problem (5.1).

Exercise 5.2. Consider the problem

$$\begin{array}{ll} \underset{x\in\Re^2}{\text{minimize}} & x_1^2 + 2x_2^2 \\ \text{subject to} & x_1 + x_2 - 1 = 0. \end{array}$$

(a) Find a point satisfying the KKT conditions. Verify that it is indeed an optimal point.

(b) Repeat Part (a) with the objective replaced by $x_1^3 + x_2^3$.

Exercise 5.3.* Write a Matlab function that will compute $c(x)$ and $J(x)$ for the constraint function

$$c(x) = x_1 + x_2 - x_1 x_2 - \frac{3}{2}.$$

Use your function to find $c(x)$ and $J(x)$ at $x = (.1,\, -.5)^T$, $x = (.5,\, -1)^T$ and $x = (1.18249728,\, -1.73976692)^T$. At each of these points, discuss the optimality of the constrained minimization problem:

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{minimize}} & e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1) \\ \text{subject to} & x_1 + x_2 - x_1x_2 - \frac{3}{2} = 0. \end{array}$$

Exercise 5.4.* The m-file newbat.m, implementing a NEWton with BAckTracking for the problem $F(x) = 0$, can be found on the class web page.

(a) Starting at $x_0 = (2,\, \frac{1}{2})^T$, $\lambda_0 = 0$, use the implementation to solve the problem in Exercise 5.3.

(b) Repeat part (a), but change the constraint to $4x_1 - x_2 - 6 = 0$.

(c) Repeat part (b), but start at $x_0 = (1,\, -2)^T$.