Exercise 4.1. Let \( f(x) \) denote a convex continuously differentiable function. Show that if a stationary point \( x^* \) exists, then \( f(x^*) \) is a global minimum of \( f \). Also show that if \( f(x) \) is actually strictly convex, then \( x^* \) is the unique global minimum. Why can uniqueness be lost if the function is not strictly convex? Draw a picture of such a situation when \( f: \mathbb{R} \rightarrow \mathbb{R} \).

If \( x^* \) is a stationary point, then, letting \( x = x^* \), \( f(y) \geq f(x^*) + f'(x^*)(y-x^*) = f(x^*) \) for all \( y \), so \( f(x^*) \) is a global minimum of \( f \).

If \( f(x) \) is strictly convex, \( f(y) > f(x^*) + f'(x^*)(y-x^*) = f(x^*) \) for all \( y \). Assume for contradiction that there exists another global minimizer \( \hat{x} \neq x^* \). But then \( f(\hat{x}) > f(x^*) \), so \( \hat{x} \) cannot be a global minimizer. So \( x^* \) is the unique global minimizer.

Exercise 4.2. This problem requires modifying the Hessian to produce a descent direction. Consider the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.
\]

(a) Derive the gradient \( g(x) \) and Hessian \( H(x) \) of \( f(x) \).

\[
g(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \cos x_3 - e^{x_2} x_3^2 \\ -x_2^2 \sin x_3 - 2e^{x_2} x_3 + 4 \end{pmatrix}, \quad H(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 \cos x_3 - e^{x_2} x_3^2 & 0 \\ 0 & -2x_2 \sin x_3 - 2e^{x_2} x_3 & -x_2 \cos x_3 - 2e^{x_2} \end{pmatrix}
\]

(b) Compute the spectral decomposition of \( H(x) \) at \( \bar{x} = (0, 1, 0)^T \).

Since \( H(\bar{x}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -(1 + 2e) \end{pmatrix} \) is a diagonal matrix, simply let \( V = I \). Then \( H(\bar{x}) = I \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -(1 + 2e) \end{pmatrix} I^T \).

(c) Compute the “pure” Newton direction \( p^N \) at \( \bar{x} \). Is \( p^N \) a descent direction?

Solving \( H(\bar{x})p^N = -g(\bar{x}) = -\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \) gives \( p^N = \begin{pmatrix} 0 \\ -1 \\ \frac{4}{1+2e} \end{pmatrix} \). Since \( g^T p^N = -2 + \frac{16}{1+2e} \approx 0.4858 > 0 \), \( p^N \) is not a descent direction.

(d) Compute the modified Newton direction \( p^M \) at the same point using the eigenvalue reflection technique (the better of the two approaches we discussed in class). Find the directional derivative along \( p^M \) at \( \bar{x} \).

\[
B(\bar{x}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 + 2e \end{pmatrix}. \quad \text{Solving } B(\bar{x})p^M = -g(\bar{x}) = -\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \text{ gives } p^M = \begin{pmatrix} 0 \\ -\frac{1}{4} \\ -\frac{1}{1+2e} \end{pmatrix}.
\]

The directional derivative is \( g^T \frac{p^M}{||p^M||_2} = \frac{-2 -16/(1+2e)}{\sqrt{1+16/(1+2e)^2}} \approx -3.810 \).

(e) Find a direction of negative curvature at \( \bar{x} \). Verify your result numerically.

We want a \( p \) such that \( p^T H(\bar{x})p < 0 \). One possible direction is \( p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \).
Exercise 4.3. Write a MATLAB function newton.m that implements a modified Newton with a backtracking line search. (This is fairly simple modification of the routine steepest.m that you wrote for the previous homework.)

Now, do the following with the implementation:

(a) Starting at $x_0 = (0, -1)^T$, apply the modified Newton method to Rosenbrock’s function

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2,$$

which has a unique minimizer at $x^* = (1, 1)^T$.

(b) Minimize the function

$$f(x) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4,$$

starting at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$. The minimizer lies at $x^* = (0, 0, 0)^T$. Discuss the differences between this run and that of part (a).

In each case, verify that the point you find is a local minimizer.

See the TA for the solution.