Exercise 3.1. Let $H$ be a symmetric matrix with spectral decomposition $H = VDV^T$.

(a) Show that an eigenvector $v$ associated with a positive eigenvalue $\lambda$ satisfies $v^T Hv > 0$.

(b) Write down the inverse of $H$ in terms of $V$ and $D$.

(c) If $r$ is a positive integer, give an expression for $H^r$ in terms of $D$ and $V$. If $H$ is positive definite, find a matrix $B$ such that $H = B^2 = BB$ ($B$ is the “square root” of $H$).

(d) Let $\alpha$ denote a scalar such that the matrix $H - \alpha I$ is nonsingular. If $\psi(\alpha)$ is the univariate function $\psi(\alpha) = u^T (H - \alpha I)^{-1} u$, where $u$ is a nonzero vector, find $\psi'(\alpha)$.

Exercise 3.2. Given each of the following cases of a gradient $g(\bar{x})$ and Hessian $H(\bar{x})$ defined at a point $\bar{x}$, discuss the optimality of $\bar{x}$. (Do NOT use MATLAB. You may need to know how to compute eigenvalues by hand in your next exam.)

(i) $g(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $H(\bar{x}) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

(ii) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $H(\bar{x}) = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

(iii) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $H(\bar{x}) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

(iv) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $H(\bar{x}) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$.

(v) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $H(\bar{x}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Exercise 3.3. Write a MATLAB function with specification $[f, g, H] = \text{ex33}(x)$ that computes $f(x), g(x)$ and $H(x)$ for the function

$$f(x) = e^{x_3}x_1^2 + 2x_2^2 + x_3^2 \cos x_1$$

at any point $x$. Use your function to compute $f(x), g(x)$, and $H(x)$ at $x = (0, 0, 0)^T$ and $x = (-1, 2, -2)^T$. In each case, compute the spectral decomposition of the Hessian matrix and indicate if the necessary and sufficient conditions for unconstrained local minization are satisfied.

Exercise 3.4. Let $q(x), x \in \mathbb{R}^n$, be the quadratic function $q(x) = c^T x + \frac{1}{2} x^T H x$, where $H$ is symmetric.

(a) Write down an expression for $\nabla q(x)$ in terms of $c, H$ and $x$.

(b) Given an arbitrary point $x_0$ and a direction $p$, write down the Taylor-series expansion of $q(x_0 + p)$.

(c) For this part, consider $q(x)$ such that $H$ is positive definite. If $p$ is a direction such that $\nabla q(x_0)^T p < 0$, show that there exists a positive minimizer $\alpha^*$ of $q(x_0 + \alpha p)$. Derive a closed-form expression for $\alpha^*$.

Exercise 3.5. Write a MATLAB m-file steepest.m that implements the method of steepest descent with a backtracking line search. Your function must include the following features.

- Use $\mu = \frac{1}{4}$ to define the sufficient-decrease criterion in the backtracking algorithm.
- The minimization must be terminated when either $\|g(x_k)\| \leq 10^{-5}$ or 75 iterations are performed. Any MATLAB “while” loop must include a test that will terminate the loop if it is executed more than 20 times.

Use steepest.m to find a minimizer of the function

$$f(x) = e^{x_3}x_1^2 + 2x_2^2 + x_3^2 \cos x_1,$$

starting at the point $(-1, 1, 1)^T$. Next, minimize the function (first write a MATLAB function as in Exercise 3.3)

$$f(x) = x_1 + x_2 + x_3 + x_4 + x_1^2 + x_2^2 + 10^{-1}x_3^2 + 10^{-3}x_4^2,$$

starting at the point $(-1, 0, 1, 1)^T$. Compare the two runs. Can you explain why steepest descent behaves like this?