MATH 171B: Numerical Optimization: Nonlinear Problems

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Homework Assignment #3
Due (See Class Webpage for Due Date)

The starred exercises are those that require the use of MATLAB. Remember, you must do the MATLAB problems to get credit for the homework.

Exercise 3.1. Let H be a symmetric matrix with spectral decomposition $H = VDV^T$.

(a) Show that an eigenvector v associated with a positive eigenvalue λ satisfies $v^T Hv > 0$.
(b) Write down the inverse of H in terms of V and D.
(c) If r is a positive integer, give an expression for H^r in terms of D and V. If H is positive definite, find a matrix B such that $H = B^2 = BB$ (B is the "square root" of H).
(d) Let α denote a scalar such that the matrix $H - \alpha I$ is nonsingular. If $\psi(\alpha)$ is the univariate function $\psi(\alpha) = u^T(H - \alpha I)^{-1}u$, where u is a nonzero vector, find $\psi'(\alpha)$.

Exercise 3.2. Given each of the following cases of a gradient $g(\bar{x})$ and Hessian $H(\bar{x})$ defined at a point $\bar{x}$, discuss the optimality of $\bar{x}$. (Do NOT use MATLAB. You may need to know how to compute eigenvalues by hand in your next exam.)

(i) $\quad g(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$
   $\qquad$ **(ii)** $\quad g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}.$

(iii) $\quad g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$
   $\qquad$ **(iv)** $\quad g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$

(v) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$

Exercise 3.3.[*] Write a MATLAB function with specification `[f,g,H] = ex33(x)` that computes $f(x)$, $g(x)$ and $H(x)$ for the function
$$f(x) = e^{x_3}x_1^2 + 2x_2^2 + x_3^2 \cos x_1$$
at any point x. Use your function to compute $f(x)$, $g(x)$, and $H(x)$ at $x = (0,\ 0,\ 0)^T$ and $x = (-1,\ 2,\ -2)^T$. In each case, compute the spectral decomposition of the Hessian matrix and indicate if the necessary and sufficient conditions for unconstrained local minimization are satisfied.

Exercise 3.4. Let $q(x)$, $x \in \Re^n$, be the quadratic function $q(x) = c^T x + \frac{1}{2}x^T Hx$, where H is symmetric.

(a) Write down an expression for $\nabla q(x)$ in terms of c, H and x.
(b) Given an arbitrary point x_0 and a direction p, write down the Taylor-series expansion of $q(x_0 + p)$.
(c) For this part, consider $q(x)$ such that H is positive definite. If p is a direction such that $\nabla q(x_0)^T p < 0$, show that there exists a *positive* minimizer α^* of $q(x_0 + \alpha p)$. Derive a closed-form expression for α^*.

Exercise 3.5.[*] Write a MATLAB m-file `steepest.m` that implements the method of steepest descent with a backtracking line search. Your function *must* include the following features.

- Use $\mu = \frac{1}{4}$ to define the sufficient-decrease criterion in the backtracking algorithm.
- The minimization must be terminated when either $\|g(x_k)\| \leq 10^{-5}$ or 75 iterations are performed. Any MATLAB "while" loop must include a test that will terminate the loop if it is executed more than 20 times.

Use `steepest.m` to find a minimizer of the function
$$f(x) = e^{x_3}x_1^2 + 2x_2^2 + x_3^2 \cos x_1,$$
starting at the point $(-1,\ 1,\ 1)^T$. Next, minimize the function (first write a MATLAB function as in Exercise 3.3)
$$f(x) = x_1 + x_2 + x_3 + x_4 + x_1^2 + x_2^2 + 10^{-1}x_3^2 + 10^{-3}x_4^2,$$
starting at the point $(-1,\ 0,\ 1,\ 1)^T$. Compare the two runs. Can you explain why steepest descent behaves like this?