MATH 171B: Numerical Optimization: Nonlinear Problems

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Homework Assignment #1
Due (See Class Webpage for Due Date)

Our goal in this homework is to review some basic concepts from linear algebra and from calculus of real and multivariate functions. Key tools throughout the course will be Taylor series and the Taylor Remainder Theorem. We will do a couple of simple implementation problems to start us off using MATLAB. *This homework covers mainly material from Chapter 1 of the textbook, and is mainly a review of things you (should) already know.*

The starred exercises require use of MATLAB*; you MUST do the* MATLAB *problems to get credit for the homework.*

Exercise 1.1. If x is an eigenvector of A, show that βx is also an eigenvector for any $\beta \neq 0$. What is the associated eigenvalue? Use this result to show that the unit vector $x/\|x\|$ formed from an eigenvector x is also an eigenvector of A corresponding to the same eigenvalue as that of x.

Exercise 1.2. Let $(x,y) : V \mapsto \mathbb{R}$ be an inner-product on a vector space V with associated scalar field $\mathbb{R}$. We know that (x,y) must satisfy the three properties of an inner-product:

1. $(x,x) \geq 0, \quad \forall x \in V, \quad (x,x) = 0$ iff $x = 0$.
2. $(x,y) = (y,x), \quad \forall x,y \in V$.
3. $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z), \quad \forall \alpha,\beta \in \mathbb{R}, \quad \forall x,y,z \in V$.

Use these three properties to show that the induced norm $\|x\| = (x,x)^{1/2}$ satisfies the three properties of a norm:

1. $\|\alpha x\| = |\alpha| \, \|x\|, \quad \forall \alpha \in \mathbb{R}, \quad \forall x \in V$.
2. $\|x\| \geq 0, \quad \forall x \in V, \quad \|x\| = 0$ iff $x = 0$.
3. $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x,y \in V$.

Hint: Showing the first two properties is very easy; to show the last property (triangle inequality), assume the Cauchy-Schwarz inequality holds: $|(x,y)| \leq \|x\| \, \|y\|$.

Exercise 1.3. Let $F(x)$ denote a twice-differentiable function of one variable. Assuming only the mean-value theorem of integral calculus: $F(b) = F(a) + \int_a^b F'(t)\,dt$, derive the following variants of the Taylor-series expansion with integral remainder:

(a) $F(x+h) = F(x) + \int_x^{x+h} F'(t)\,dt$.

(b) $F(x+h) = F(x) + h \int_0^1 F'(x+\xi h)\,d\xi$.

(c) $F(x+h) = F(x) + hF'(x) + h \int_0^1 [F'(x+\xi h) - F'(x)]\,d\xi$.

(d) $F(x+h) = F(x) + hF'(x) + h^2 \int_0^1 F''(x+\xi h)(1-\xi)\,d\xi$. (Hint: Try expanding $F'(x+h)$ using a formula like part (b) and then differentiate with respect to h using the chain rule.)

Exercise 1.4. Find the gradient vector $F(x) = \nabla f(x)$ of the following functions, and then find the Jacobian matrix of $F(x)$. (The Jacobian matrix of $F(x) = \nabla f(x)$ is the same as the Hessian matrix $\nabla^2 f(x)$ of $f(x)$).

(a) $f(x) = 2\big(x_2 - x_1^2\big)^2 + \big(x_1 - 3\big)^2$.

(b) $f(x) = (2x_1 + x_2)^2 + 4(x_2 - x_3)^4$.

Exercise 1.5. Find $f'(x)$, $\nabla f(x)$ and $\nabla^2 f(x)$ for the following functions of n variables.

(a) $f(x) = \frac{1}{2}x^T H x$, where H is an $n \times n$ constant matrix.

(b) $f(x) = b^T A x - \frac{1}{2}x^T A^T A x$, where A is an $m \times n$ constant matrix and b is a constant m-vector.

(c) $f(x) = \|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$.

Exercise 1.6.* Create a MATLAB m-file of the form:

```
function [F,J] = D(x)
  F = [ a ; b ];
  J = [ c d ; e f ];
```

where the expressions for a, b, c, d, e, f are chosen so that the function returns the 2×1-vector-valued function $F(x)$ and the 2×2 Jacobian matrix $J(x)$ for the function $F(x)$ from part (a) of Problem 1.4. Use this m-file to compute F and J at $x = (1,\ 0)^T$; and $x = (1,\ 1)^T$. Capture the output from the computation and turn it in with the homework.