Math 171B: Mathematical Programming

Instructor: Michael Holst

Spring Quarter 2005

Homework Assignment #3
Due Friday April 29, 2005

The starred exercises are those that require the use of Matlab. *Remember, you must do the* Matlab *problems to get credit for the homework.*

Exercise 3.1. Let H be a symmetric matrix with spectral decomposition $H = VDV^T$.

(a) Show that an eigenvector v associated with a positive eigenvalue λ satisfies $v^THv > 0$.
(b) Write down the inverse of H in terms of V and D.
(c) If r is a positive integer, give an expression for H^r in terms of D and V. If H is positive definite, find a matrix B such that $H = B^2 = BB$ (B is the "square root" of H).
(d) Let α denote a scalar such that the matrix $H - \alpha I$ is nonsingular. If $\psi(\alpha)$ is the univariate function $\psi(\alpha) = u^T(H - \alpha I)^{-1}u$, where u is a nonzero vector, find $\psi'(\alpha)$.

Exercise 3.2. Given each of the following cases of a gradient $g(\bar{x})$ and Hessian $H(\bar{x})$ defined at a point $\bar{x}$, discuss the optimality of $\bar{x}$. (Do NOT use Matlab. You may need to know how to compute eigenvalues by hand in your next exam.)

(i) $$g(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

(ii) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}.$$

(iii) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

(iv) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

(v) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Exercise 3.3.* Write a Matlab function with specification `[f,g,H] = ex33(x)` that computes $f(x)$, $g(x)$ and $H(x)$ for the function

$$f(x) = e^{x_3}x_1^2 + 2x_2^2 + x_3^2 \cos x_1$$

at any point x. Use your function to compute $f(x)$, $g(x)$, and $H(x)$ at $x = (0,\ 0,\ 0)^T$ and $x = (-1,\ 2,\ -2)^T$. In each case, compute the spectral decomposition of the Hessian matrix and indicate if the necessary and sufficient conditions for unconstrained local minimization are satisfied.

Exercise 3.4. Let $q(x)$, $x \in I\!R^n$, be the quadratic function $q(x) = c^Tx + \frac{1}{2}x^THx$, where H is symmetric.

(a) Write down an expression for $\nabla q(x)$ in terms of c, H and x.
(b) Given an arbitrary point x_0 and a direction p, write down the Taylor-series expansion of $q(x_0 + p)$.
(c) For this part, consider $q(x)$ such that H is positive definite. If p is a direction such that $\nabla q(x_0)^Tp < 0$, show that there exists a *positive* minimizer α^* of $q(x_0 + \alpha p)$. Derive a closed-form expression for α^*.

Exercise 3.5.* Write a Matlab m-file `steepest.m` that implements the method of steepest descent with a backtracking line search. Your function *must* include the following features.

- Use $\mu = \frac{1}{4}$ to define the sufficient-decrease criterion in the backtracking algorithm.
- The minimization must be terminated when either $\|g(x_k)\| \le 10^{-5}$ or 75 iterations are performed. Any Matlab "`while`" loop must include a test that will terminate the loop if it is executed more than 20 times.

Use `steepest.m` to find a minimizer of the function

$$f(x) = e^{x_3}x_1^2 + 2x_2^2 + x_3^2 \cos x_1,$$

starting at the point $(-1,\ 1,\ 1)^T$. Next, minimize the function (first write a Matlab function as in Exercise 3.3)

$$f(x) = x_1 + x_2 + x_3 + x_4 + x_1^2 + x_2^2 + 10^{-1}x_3^2 + 10^{-3}x_4^2,$$

starting at the point $(-1,\ 0,\ 1,\ 1)^T$. Compare the two runs. Can you explain why steepest descent behaves like this?