MATH 170B HW4

6.1

#2 Proof: ① By Thm I, Lf is unique, and Langrange form is that unique polynomial, thus

$$Lf = \sum f(x_i) l_i .$$

② Show L is linear.

$$L(af+bg) = \cancel{aLf+bLg}$$
$$\sum (af+bg)(x_i) l_i = a\sum f(x_i) l_i + b\sum g(x_i) l_i$$
$$= aLf + bLg.$$

□.

#4. Proof: By Thm I, Lq is the unique polynomial at order n most s.t. $Lq(x_i) = q(x_i)$.

And q itself is $q(x_i) = q(x_i)$,

thus, q is the unique polynomial, i.e. $Lq = q$.

□.

#5. Proof: Consider $f(x) = 1$, then

$$Lf = \sum_{i=0}^{n} f(x_i) l_i(x) = \sum_{i=0}^{n} l_i(x).$$

By #4, $Lf = f$, thus $\sum_{i=0}^{n} l_i(x) = 1$.

□

#6 proof.

$$\sum_{i=0}^{n} [f(x)-f(x_i)]l_i(x) = f(x)\sum_{i=0}^{n} l_i(x) - \sum_{i=0}^{n} f(x_i)\, l_i(x) \quad \cdots (1)$$

then by #2 and #5

$$(1) = f(x) - p(x).$$

$\square$.

#21

x	2	0	3
$f(x)$	11	7	28

① Lagrange form:

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-3)}{(2-0)(2-3)} = -\frac{1}{2}x(x-3)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-2)(x-3)}{(0-2)(0-3)} = \frac{1}{6}(x-2)(x-3)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-2)(x-0)}{(3-2)(3-0)} = \frac{1}{3}(x-2)x$$

$$\Rightarrow p(x) = -\frac{11}{2}x(x-3) + \frac{7}{6}(x-2)(x-3) + \frac{28}{3}x(x-2).$$

② Newton's form:

$$p(x) = c_0 + c_1(x-2) + c_2(x-2)(x-0)$$

Then solve

$$\begin{cases} p(x_0) = 11 \\ p(x_1) = 7 \\ p(x_2) = 28 \end{cases} \Rightarrow \begin{cases} c_0 = 11 \\ c_0 + c_1(0-2) = 7 \\ c_0 + c_1(3-2) + c_2(3-2)(3-0) = 28 \end{cases}$$

$$\Rightarrow \begin{cases} c_0 = 11 \\ c_1 = 2 \\ c_2 = 5 \end{cases} \Rightarrow p(x) = 11 + 2(x-2) + 5(x-2)x.$$

#22. The same as #21.

6.2 #5. proof: According to (10) on page 329,

$\sum_{i=0}^{n} p[x_0, x_1, \cdots, x_i] \prod_{j=0}^{i-1} (x-x_j)$ is the newton's form.

By #4 in 6.1, $p(x) = Lp = \sum_{i=0}^{n} p[x_0 \cdots x_i] \prod_{j=0}^{i-1} (x-x_j)$. ◻.

#8. proof: By Thm 1, Lagrange form = Newton's form ◻.

#24.

x	4	2	0	3
$f(x)$	63	11	7	28

$p(x) = c_0 + c_1(x-4) + c_2(x-4)(x-2) + c_3(x-4)(x-2)x$

	x	$f(x)$			
x_0	4	63	26	6	1
x_1	2	11	2	5	
x_2	0	7	7		
x_3	3	28			

upper trianglar table

$c_0 = 63$

$c_1 = 26$

$c_2 = 6$

$c_3 = 1$.

	x	$f(x)$			
x_0	4	63			
			26		
x_1	2	11		6	
			2		
x_2	0	7			1
x_3	3	28	7	5	

Lower trianglar table

6.3 #1.

x	0	1	2
$p(x)$	2	-4	44
$p'(x)$	-9	4	X

Follow these arrows in data, one has

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^2(x-1) + c_4 x^2 (x-1)^2$$

	x	$f(x)$				
			$p'(0)=-9$	3	7	5
x_0	0	2				
x_0	0	2	-6	10	17	
x_1	1	-4	$p'(1)=4$	44		
x_1	1	-4	48			
x_2	2	44				

upper

$\Rightarrow$ $c_0 = 2$, $c_1 = -9$, $c_2 = 3$, $c_3 = 7$, $c_4 = 5$

	x	$f(x)$				
x_0	0	2	-9			
x_0	0	2	-6	3		
x_1	1	-4				
x_1	1	-4	4	10	7	
x_2	2	44	48	44	17	5

lower

See 6.8.

#4 Use Induction.

① $\{P_0\}$ is obviously linear indep.

② Assume $\{P_0, \cdots, P_k\}$ is linear indep.

Then consider $\{P_0, \cdots, P_{k+1}\}$

$\sum_{i=0}^{k+1} a_i P_i = 0$, since only P_{k+1} has term x^{k+1}, so $a_{k+1} = 0$

$\Rightarrow \sum_{i=0}^{k} a_i P_i = 0 - a_{k+1} P_{k+1} = 0$, by assumption, $a_i = 0, i = 0, \cdots, k$

$\Rightarrow a_i = 0, \cdots, k+1$. D.

#5 Show $\langle f, g\rangle = \sum_{i=1}^{n} \langle f, u_i\rangle \langle g, u_i\rangle$, $\{u_1, \cdots, u_n\}$ is orthonormal set.

Proof: ~~$\langle f, g\rangle$~~ Since $f = \sum_{i=1}^{n} \alpha_i u_i$, $g = \sum_{i=1}^{n} \beta_i u_i$,

$$\langle f, g\rangle = \langle \sum_{i=1}^{n} \alpha_i u_i, \sum_{j=1}^{n} \beta_j u_j\rangle \text{ by orthogonality}$$

$$= \sum_{i=1}^{n} \langle \alpha_i u_i, \beta_i u_i\rangle, \text{ by } \|u_i\| = 1$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i$$

$$\sum_{i=1}^{n} \langle f, u_i\rangle\langle g, u_i\rangle = \sum_{i=1}^{n} \langle \sum_{j=1}^{n} \alpha_j u_j, u_i\rangle \langle \sum_{j=1}^{n} \beta_j u_j, u_i\rangle$$

$$= \sum_{i=1}^{n} \langle \alpha_i u_i, u_i\rangle\langle \beta_i u_i, u_i\rangle$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i.$$

$\square$.

#15 Proof: Let $\{u_1, \cdots, u_n\}$ be a orthogonal set, and all $u_i \neq 0$.

Consider $\sum_{i=1}^{n} a_i u_i = 0$

then WLOG $\langle \sum_{i=1}^{n} a_i u_i, u_i\rangle = \langle 0, u_i\rangle = 0$

$\Rightarrow$ $a_i \langle u_i, u_i\rangle = a_i \|u_i\|^2 = 0$, since $u_i \neq 0$

$\Rightarrow$ $a_i = 0$, $i = 1, \cdots, n$.

Therefore, $\{u_1, \cdots, u_n\}$ is linear indep set.

$\square$

#16 Proof: Let $Af_1 = \lambda_1 f_1$

$Af_2 = \lambda_2 f_2$, $\lambda_1 \neq \lambda_2$

then $\langle f_1, Af_2 \rangle = \boxed{\langle Af_1, f_2 \rangle = \langle \lambda_1 f_1, f_2 \rangle} = \lambda_1 \langle f_1, f_2 \rangle$

by self adjoint

$= \langle f_1, \lambda_2 f_2 \rangle = \lambda_2 \langle f_1, f_2 \rangle$.

Finally, we have $\lambda_2 \langle f_1, f_2 \rangle = \lambda_1 \langle f_1, f_2 \rangle$, since $\lambda_1 \neq \lambda_2$

$\Rightarrow \langle f_1, f_2 \rangle = 0$, i.e $f_1 \perp f_2$.

$\square$.

#18. Proof: Thm 7

① $P_n(\alpha f + \beta g) = \sum_{i=1}^{n} \langle \alpha f + \beta g, u_i \rangle u_i = \alpha \sum_{i=1}^{n} \langle f, u_i \rangle u_i + \beta \sum_{i=1}^{n} \langle g, u_i \rangle u_i$

$= \alpha P_n f + \beta P_n g$, P_n is Linear.

Then show P_n is "onto".

$\forall g \in U_n$, ~~one can~~ i.e. $g = \sum_{i=1}^{n} a_i u_i$ and $g \in E$ also.

then consider $P_n g = \sum_{i=1}^{n} \langle g, u_i \rangle u_i = \sum_{i=1}^{n} \langle \sum_{j=1}^{n} a_j u_j, u_i \rangle u_i$

$= \sum_{i=1}^{n} a_i u_i = g$,

thus P_n is surjective.

② Show $P_n^2 = P_n$

In ①, we already showed if $g \in U_n$, then $P_n g = g$,

thus $\cancel{P_n (P_n\ P_n g = P_n(P_n g) = P_n g}$

For $f \in E$, $P_n^2 f = P_n(P_n f) = P_n f$. , i.e. $P_n^2 = P_n$.

③ Let $f \in E$, $P_n f = \sum_{i=1}^{n} \langle f, u_i \rangle u_i \in U_n$, $\forall g \in U_n$, $g = \sum_{i=1}^{n} \beta_i u_i$.

Then $\langle f - P_n f, g \rangle = \langle f - \sum_{i=1}^{n} \langle f, u_i \rangle u_i, \sum_{i=1}^{n} \beta_i u_i \rangle$

$$= \langle f, \sum_{i=1}^{n} \beta_i u_i \rangle - \langle \sum_{i=1}^{n} \langle f, u_i \rangle u_i, \sum_{j=1}^{n} \beta_j u_j \rangle$$

$$= \sum_{i=1}^{n} \langle f, u_i \rangle \beta_i - \sum_{i=1}^{n} \langle f, u_i \rangle \langle u_i, u_i \rangle \beta_i$$

Since $\|u_i\| = 1$ orthonormal

$$= \sum_{i=1}^{n} \langle f, u_i \rangle \beta_i - \sum_{i=1}^{n} \langle f, u_i \rangle \beta_i = 0.$$

Since g in arbitrary, $f - P_n f \perp U_n$.

④ Consider $\|f - P_n f\|^2 = \langle f - P_n f, f - P_n f \rangle$

$$= \langle f, f - P_n f \rangle - \underbrace{\langle P_n f, f - P_n f \rangle}_{0 \text{ by ③}}$$

$$= \langle f - g + g, f - P_n f \rangle, \quad \forall g \in U_n$$

$$= \langle f - g, f - P_n f \rangle + \underbrace{\langle g, f - P_n f \rangle}_{0 \text{ by ③}}$$

by Cauchy-schwarz $\leq \|f - g\| \, \|f - P_n f\|$.

$\Rightarrow$ Divide $\|f - P_n f\|$, $\|f - P_n f\| \leq \|f - g\|$, $\forall g \in U_n$.

i.e. $P_n f$ is best approximation.

Let $f, g \in E$

⑤ $\langle P_n f, g \rangle =$

$$\langle P_n f, g - P_n g + P_n g \rangle = \underbrace{\langle P_n f, g - P_n g \rangle}_{0 \text{ by ③}} + \langle P_n f, P_n g \rangle$$

$$= \langle P_n f - f + f, P_n g \rangle$$

$$= \underbrace{\langle P_n f - f, P_n g \rangle}_{0 \text{ by ③}} + \langle f, P_n g \rangle$$

$\square$.

#21 By Thm 5: from Example 2,

we have $P_0=1$, $P_1=X$, $P_2=X^2-\frac{1}{3}$.

then $P_3(X)=(X-a_3)P_2-b_3P_1$

where $a_3=\dfrac{\langle XP_2,P_2\rangle}{\langle P_2,P_2\rangle}=\dfrac{\int_{-1}^{1}x(x^2-\frac{1}{3})^2dx}{\int_{-1}^{1}(x^2-\frac{1}{3})^2dx}=0$

$b_3=\dfrac{\langle XP_2,P_1\rangle}{\langle P_1,P_1\rangle}=\dfrac{\int_{-1}^{1}x(x^2-\frac{1}{3})x\,dx}{\int_{-1}^{1}x^2dx}=\dfrac{4}{15}$

$\Rightarrow P_3=XP_2-\frac{4}{15}P_1=X^3-\frac{3}{5}X.$

Use the same algorithm to find P_4 and P_5.

#22. Here the space becomes $C[0,1]$,

and inner-product is $\int_0^1 fg\,dx$.

Let $P_0=1$, $P_1=X-a_1$

① Find $a_1=\dfrac{\langle XP_0,P_0\rangle}{\langle P_0,P_0\rangle}=\dfrac{\int_0^1 x\,dx}{\int_0^1 1\,dx}=\dfrac{1}{2}$

$\Rightarrow P_1=X-\frac{1}{2}$

② $P_2=(X-a_2)P_1-b_2P_0$

where $a_2=\dfrac{\langle XP_1,P_1\rangle}{\langle P_1,P_1\rangle}=\dfrac{\int_0^1 x(x-\frac{1}{2})^2dx}{\int_0^1(x-\frac{1}{2})^2dx}=\dfrac{1}{2}$

$b_2=\dfrac{\langle XP_1,P_0\rangle}{\langle P_0,P_0\rangle}=\dfrac{\int_0^1 x(x-\frac{1}{2})dx}{\int_0^1 1\,dx}=\dfrac{1}{12}$

$\Rightarrow P_2=(X-\frac{1}{2})(X-\frac{1}{2})-\frac{1}{12}=X^2-X+\frac{1}{6}$.

Similar for P_3.

□